Stability of Jackson-type queueing networks, I

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Stability of Jackson-Type Queueing Networks, I

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Abstract

This paper gives a pathwise construction of Jackson-type queueing networks allowing the derivation of stability and convergence theorems under general statistical assumptions on the driving sequences; namely, it is only assumed that the input process, the service sequences and the routing mechanism are jointly stationary and ergodic in a sense that is made precise in the paper. The main tools for these results are the subadditive ergodic theorem, which is used to derive a strong law of large numbers, and basic theorems on monotone stochastic recursive sequences. The techniques which are proposed here apply to other and more general classes of discrete event systems, like Petri nets or GSMP's. The paper also provides new results on the Jackson-type networks with i.i.d. driving sequences which were studied in the past.

Keywords: Ordered directed graph, generated ordered directed graph, switching sequence, move sequence, open Jackson-type queueing network, point processes, simple network, composition, decomposition, conservation rule, departure and throughput processes, first and second-order ergodic properties, subadditive ergodic theorem, solidarity property, stochastic recursive sequences, stationary solution, coupling-convergence, uniqueness of the stationary regime.

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Stabilité des réseaux de type Jackson, Partie I

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Abstract

Cet article propose une construction des trajectoires d'un réseau de files d'attente de type Jackson, qui permet d'obtenir les conditions de stabilité et de convergence sous des hypothèses statistiques générales (stationnaires ergodiques) sur les processus de service, de routage et d'arrivée. Les outils principaux de cette construction sont le théorème sous-additif ergodique de Kingman, qui permet d'établir une loi forte des grands nombres, et certaines propriétés de suites récurrentes stochastiques monotones. Ces techniques s'appliquent à des classes de réseaux plus générales et notamment aux réseaux de Petri stochastiques. Des résultats nouveaux sont aussi obtenus pour la classe des réseaux de Jackson avec des processus i.i.d, déjà étudiée dans le passé.

Mots Clés: Graphe orienté ordonné, suite de commutateurs, suite de transitions, réseaux de type Jackson, processus ponctuels, réseau simple, composition et décomposition, loi de conservation, théorèmes ergodiques de premier ordre et de deuxième ordre, théorème sous-additif ergodique, solidarité, suites récurrentes stochastiques, régimes stationnaires, couplage, unicité des régimes stationnaires

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1 Introduction

A class of queueing systems is often considered as well understood if its state can be constructed and its properties analyzed under general statistical assumptions, namely stationarity and ergodicity assumptions on the data of the system under consideration (see [27], [8], [20], [2], [14], [12]). Such constructions and analysis have been available for quite general classes of acyclic queueing networks (see [22], [25], [3] for instance), but only for specific classes of acyclic networks (see [6], [5], [1]). Most of these contributions are based on pathwise recursions which can be traced back to the pioneering work of Loynes [27].

To the best of our knowledge, for Jackson-type networks, the stability problem was only approached either under specific statistical assumptions (this is the case for their definition using product form theory by Jackson [21]), or under certain modifications of the service mechanism (see for instance [6] and [1] which introduce either synchronization constraints or priorities in order to analyze the network). Although some of the models focusing on the actual Jackson-type problem are rather general (see [10], [17], [18], [26] and [25] for instance), all of them require some sort of independence property or some distributional restrictions (see [18] for a partial bibliography on the matter). More generally, for this type of general assumptions, no construction of the state of the network providing ergodic theorems seems to be currently available. The object of the present paper is to make such a construction.

A first difficulty arises with the definition of such a generalized Jackson-type queueing network. The networks we consider here are characterized by the fact that service times and switching decisions are associated with stations, and not with customers. This means that the j-th service on station k takes σ_j^k units of time. In addition, when this service is completed, the leaving customer is sent to station ν_j^k (or leaves the network) and it is put at the end of the queue on this station. This numbering mechanism has to be opposed to what happens in Kelly-type networks where routing (and service times) are associated with input customers (e.g. an arriving customer has a predefined route and predefined service requirements at each of the stations of its route, all of which are known upon its arrival). This distinction is essential for the purposes of the present paper as the first type of numbering preserves various basic monotonicity properties as shown in [17], [18] and [30].

The second difficulty lies in the construction of state variables amenable to some sort of stochastic recurrence equation which satisfies a first-order ergodic theorem (i.e. a SLLN). These variables will be referred to as first-order state variables. The possibility of defining such state variables is obtained from recursive equations which were derived for a class of Stochastic Petri net which contains Jackson networks ([4]). The understanding of the appropriate stationarity and ergodicity assumptions to be made on service and routing sequences comes from graph theoretic considerations, and in particular from a composition rule explaining the geometry of routing in this class of networks. This graph theoretic construction is tantamount to the construction of the reachability graph of the underlying Petri net, and although it is not essential to understand the main ergodic theorems, we decided to put it in the second section, simply because it provides to an in-depth understanding on the pathwise dynamics of such networks. In particular, the so-called decomposition property, which is established there, reveals the right ergodicity assumptions to be made on these 'driving' sequences. The first-order ergodic theorem follows from the subadditive property satisfied by the time to clear the system of its workload after the last epoch of an interrupted arrival point process (§4). This technique follows that of [5]. The time to clear the workload is an adequate variable for getting a SLLN under rather general assumptions since this is true for event graphs, for Jackson networks, and for stochastic Petri nets with general topology [7].

The third difficulty comes from the search for increments of the first-order state variables which satisfy some stochastic recurrence equation with appropriate monotonicity properties and a second-order ergodic result of the Loynes-type (e.g. coupling with a stationary ergodic regime or simply weak convergence to such a regime). These second-order state variables are introduced in §3 (the so-called Γ process), following ideas developed in [18]. The relation between the finiteness of the second-order state variables and the constants that show up in the SLLN is investigated in § 5. This gives the stability threshold ensuring the finiteness of queue length and the like. The stochastic recursion that these second-order variables satisfy is investigated in §6 and used in §7 for proving certain coupling convergence results and uniqueness results.

Besides the theoretical interest of this construction, several new results or new proofs of known results can be obtained for various models. For instance, we can always construct a minimal stationary regime (see §6). In the particular case when routing and services are i.i.d., the distinction between service associated with stations and service associated with customers vanishes as both coincide in law. So, when restricted to the i.i.d. case, our results show that the Cramer-type conditions considered in [10] on the distribution functions of the service times can be relaxed (see [18]; this fact was observed independently in [15]). Similarly, whenever the switching decisions are i.i.d, we show the following generalization of results in [18]: there is a unique stationary regime which is reached with coupling, under general assumptions on the arrival and service processes.

2 Ordered Directed Graphs.

2.1 Basic Definitions.

Let K and φ be two positive integers.

Definition 1 *The finite sequence of integers $r = (r_1, \dots, r_\varphi, r_{\varphi+1})$ is a K-node route with length φ if $1 \le r_i \le K$ for all $i = 1, \dots, \varphi$ and $1 \le r_{\varphi+1} \le K + 1$.*

Definition 2 *The route r is admissible if $r_{\varphi+1} \in \{r_1, K+1\}$ and successful if $r_{\varphi+1} = K + 1$. If $r_{\varphi+1} = r_1$, r will also be called a circuit. This circuit will be said to be simple if it contains no other circuit.*

Consider an admissible route r with length φ. For each $k = 1, \dots, K$ let

$$\varphi^k = \sharp\{i : 1 \le i \le \varphi, r_i = k\}. \tag{1}$$

With route r and node $k \in \{1, \dots, K\}$, we associate a $\{1, \dots, K, K+1\}$-valued sequence ν^k which is defined as follows:

- if $\varphi^k = 0$ then $\nu^k = \emptyset$ (i.e. ν^k is the empty sequence);
- if $\varphi^k > 0$ then
  - Consider the auxiliary sequence $\{q_n^k\}_{n=1}^{\varphi^k}$ defined by $q_1^k = \min\{i \ge 1 : r_i = k\}$ and for $n = 1, \dots, \varphi^k - 1$ by $q_{n+1}^k = \min\{i \ge q_n^k + 1 : r_i = k\}$;
  - For $n = 1, \dots, \varphi^k$ let $\nu_n^k = r_{q_n^k+1}$.

Then ν^k is the sequence $\nu^k = \{\nu_1^k, \dots, \nu_{\varphi^k}^k\}$. We shall say that ν^k is the *switching sequence of node k* associated with route r, and that $\nu = \{\nu^k\}_{k=1}^K$ is the *switching sequence associated with route r.*

For each $k = 1, \dots, K$, $l = 1, \dots, K, K+1$ let

$$\varphi^{k,l} = \sharp\{i : r_i = k,\ r_{i+1} = l\} \tag{2}$$

denote the number of arcs from node k to node l along route r. Our definitions imply the following relations:

$$\varphi^{k,l} = \sharp\{j : 1 \le j \le \varphi^k; \nu_j^k = l\}, \quad \varphi^{k,K+1} = I(r_\varphi = k; r_{\varphi+1} = K+1); \tag{3}$$

$$\varphi^l = \sum_{k=1}^{K} \varphi^{k,l} + I(r_1 = l), \quad \varphi^k = \sum_{l=1}^{K+1} \varphi^{k,l}; \tag{4}$$

for all $k = 1, \dots, K$, $l = 1, \dots, K+1$.

2.2 Composition of Switching Sequences

Let N be a positive integer, $R = (r(1), \ldots, r(N))$ be a sequence of successful routes with lengths $\varphi(1), \ldots, \varphi(N)$ and with switching sequences $\nu(1), \ldots, \nu(N)$, respectively. More generally, we shall add the argument (n) to a variable v defined in the preceding section in order to denote the variable v associated with route $r(n)$: for instance, $\varphi^k(n)$ denotes the number of visits of $r(n)$ to node k.

For each $k = 1, \ldots, K$, let $\nu^k[N]$ denote the sequence:

$$\nu^k[N] = \{\nu_1^k(1), \ldots, \nu_{\varphi^k(1)}^k(1), \ldots, \nu_1^k(N), \ldots, \nu_{\varphi^k(N)}^k(N)\} \tag{5}$$

(where $\nu^k[N] = \emptyset$ if $\varphi^k(1) = \ldots = \varphi^k(N) = 0$) and let $\nu[N] = \{\nu^k[N]\}_{k=1}^K$.

We shall say that $\nu[N]$ is the *switching sequence* associated with the sequence of routes R and $\nu[N]$ is the *composition* of the sequences $\nu(1), \ldots, \nu(N)$. Thus the composition of switching sequences is merely their concatenation.

2.3 Ordered Directed Graphs

Consider a directed graph $G = (\mathcal{N}, \mathcal{A})$, with set of nodes $\mathcal{N} = \{1, 2, \ldots, K+1\}$ and with set of arcs $\mathcal{A}$. Since the edges are directed, we can speak about "input" and "output" arcs on each node. For $k = 1, \ldots, K+1$, we denote I^k the set of input arcs and O^k the set of output arcs on node k. We assume that

$$O^{K+1} = \emptyset. \tag{6}$$

For $k = 1, \ldots, K$, let

$$c^k = \sharp\{I^k\}, \quad d^k = \sharp\{O^k\}. \tag{7}$$

Definition 3 *The directed graph G is an ordered directed graph (O.D.G.) if for each node, the output arcs are labelled in a totally ordered way: namely, for all k, the arcs originating from k are labelled $p_1^k, \ldots, p_{d^k}^k \in \mathbb{N}$, where $p_j^k < p_{j+1}^k$, for all j.*

We are now ready to define the *paths* of an O.D.G. G. The path $l^k = (l^k(1), l^k(2), \ldots)$ originating from node $k = 1, \ldots, K$, is defined by the following procedure:

Procedure 1

$G(1) := G$; $n(1) := k$; $t := 1$;

while $d^{n(t)} > 0$ do

begin

- $l^k(t) := n(t)$;
- $t := t + 1$;
- $n(t) :=$ the end node of the arc of $G(t)$ which originates from $n(t)$ and with the smallest index (label);
- $G(t) :=$ the O.D.G. obtained from $G(t)$ by removing this arc;

end

Since the number of arcs in the original graph G is finite, the path originating from node k is a finite sequence

$$l^k = (l^k(1), \dots, l^k(m)), \tag{8}$$

where $l^k(1) = k$. The sequence of arcs that we "remove" when executing this procedure will be called the sequence of arcs associated with the path l^k.

Let G be an O.D.G. We denote V^k the sequence $(v_1^k, \dots, v_p^k)$ if the set of arcs originating from k reads $(k, v_1^k), \dots, (k, v_p^k)$, when ordered.

Definition 4 *An O.D.G. G is called a generated ordered directed graph (G.O.D.G.) if there exist an integer $N \geq 1$ and a sequence of successful routes $R = (r(1), \dots, r(N))$ such that for all k, the switching sequence $\nu^k[N]$ associated with this route coincides with V^k. In this case we say that R is a generator of G.O.D.G. G.*

Remark 1 *A G.O.D.G. may have several generators. If $R = (r(1), \dots, r(N))$ is a generator of G, then $N = c_{K+1}$. So the number N is the same for all generators. Moreover, if we denote*

$$T^k = \sharp\{i : 1 \leq i \leq N; r_1(i) = k\} \tag{9}$$

for $k = 1, \dots K$ then $T^k = d^k - c^k$ for all k, and the sequence $\{T^k\}_{k=1}^K$ is again the same for all generators (note that $N = T^1 + \dots + T^K$).

Remark 2 *Let $(i_1, \dots, i_N)$ be a permutation of $(1, \dots, N)$. If $R = (r(1), \dots, r(N))$ is a generator of G.O.D.G G, then it is not true in general that $R' = (r(i_1), \dots, r(i_N))$ is also a generator of G.*

Remark 3 *We shall often call G.O.D.G. the pair* $(G, \{T^k\}_{k=1}^K)$. *So we can consider a G.O.D.G. G as a directed graph G with N "tokens", where the tokens are located on the nodes (*T^1 *tokens on node 1,* T^2 *on node 2, etc.). Consider a fixed generator. With each token, we can associate the route originating from the node where this token is located (this is a one-to-one correspondence). Note that in general, the* route *associated with a token located on node k is different from the* path *originating from node k. Understanding this point is crucial for the following theorem.*

Theorem 1 *For each G.O.D.G.* $\{G, \{T^k\}_{k=1}^K\}$ *and for each token, there exists a generator* $\hat{R} = (\hat{r}(1), \ldots, \hat{r}(N))$ *such that route* $\hat{r}(1)$ *is associated with this token.*

The proof is forwarded to Appendix 8.1. This implies in particular that whenever $T^k > 0$, if we remove the arc with minimal index originating from node k, the resulting O.D.G. is still a G.O.D.G.

2.4 Move Sequences

For an arbitrary O.D.G. G with K nodes and for all $k = 1, \ldots, K$, let φ^k denote the number of output arcs from node k and let φ be the total number of arcs (i.e. $\varphi = \varphi^1 + \ldots + \varphi^K$). If G is a G.O.D.G. with N tokens (numbered from 1 to N), then $\varphi^k = \varphi^k(1) + \ldots + \varphi^k(N)$ for all k.

Let G be a G.O.D.G. The following procedure defines a sequence of G.O.D.G. $\{G(t)\}_{t\geq 1}$, where $G(1) = G$; all these G.O.D.G. have N tokens and $(K+1)$ nodes. This sequence is associated with a *move sequence* $\{X(t)\}_{t\geq 1}$, where $X(t)$ tells us which token to 'move' within the set of *movable tokens* in $G(t)$: by definition, a token is *movable* if and only if it is not located on node $(K+1)$. Thus $X(t)$ takes it's values in $\{0, 1, \ldots, N\}$ and

- $X(t)$ cannot take the value n if token n is not movable in $G(t)$;
- $X(t) = 0$ if no tokens are movable in $G(t)$.

Procedure 2

$G(1) := G$; $l(1) :=$ location of token $X(1)$; $t := 1$;

while $d^{l(t)} > 0$ do

begin

- $l(t) :=$ location of token $X(t)$;

- $n(t) :=$ end node of the arc of $G(t)$ originating in $l(t)$ and with minimal index:
- move token $X(t)$ to $n(t)$;
- $G(t+1) :=$ the graph obtained from $G(t)$ by removing this arc, while keeping the other arcs and the positions of all other tokens unchanged (this defines a new G.O.D.G. as a consequence of Theorem 1);
- $t := t+1$;

end

Let $\mathcal{A} = \{\mathcal{A}(t)\}$ be the sequence of arcs we remove under the sequence X and let H be the O.D.G. with nodes $1, \ldots, K+1$ and with ordered set of arcs $\mathcal{A}$ (i.e. the arcs from node k are ordered according to the total order induced by the sequence $\mathcal{A}$). Procedure 2 satisfies the following property (see Appendix 8.1 for the proof):

Theorem 2 *Let G be a G.O.D.G. and X a move sequence. For each $t \leq \varphi$, at least one token is movable in $G(t)$ and if $X(t) = n$ and token n is on node $k \neq K+1$, then $O^k(t) \neq \emptyset$. In addition $H = G$ and $\sharp(\mathcal{A}) = \varphi$.*

Let G be an O.D.G. on $\{1, \ldots, K\}$ and $N \geq 1$ be an integer. Let π be a mapping from $\{1, \ldots, N\}$ to $\{1, \ldots, K\}$, with the interpretation that $\pi(n)$ is the location of token n in G. This defines an O.D.G $(G, \{T^k\}_{k=1}^K)$ *with tokens*, where T^k still denotes the number of tokens on node k. We can also extend the notion of a move sequence to this type of graphs by the same procedure as Procedure 2 (with the only difference that the graphs $G(t)$ are not necessarily G.O.D.G. anymore).

Theorem 3 *Consider an arbitrary O.D.G. with tokens $(G, \{T^k\}_{k=1}^K)$. If a move sequence X is such that*

- *$\sharp(\mathcal{A})$ is finite;*
- *$G(\sharp(\mathcal{A}+1))$ is such that all tokens are on node $(K+1)$;*

then $(H, \{T^k\}_{k=1}^K)$ is a G.O.D.G.

3 Pathwise Construction of Open Jackson-Type Queueing Networks with Finite Input

3.1 Definitions and Notations

A *network* Σ with $K \geq 1$ nodes is a quadruplet

$$\Sigma = (N,\ T,\ \sigma,\ \nu),$$

where $N \geq 1$ is an integer,

$$T = (t(1), \ldots, t(N)) \tag{10}$$

is a real-valued sequence such that $t(1) \leq \ldots \leq t(N)$. For each $k = 0, 1, \ldots, K$,

$$\nu = \{\nu_j^k\}_{j=1}^{\infty} \tag{11}$$

is a sequence of $\{1, 2, \ldots, K+1\}$-valued numbers representing switching decisions and

$$\sigma = \{\sigma_j^k\}_{j=1}^{\infty} \tag{12}$$

is a sequence of real-valued non-negative numbers, representing service times.

These data are those associated with an open queueing network with K single-server stations, FCFS disciplines and with input sequence T. At time $t(1)-$, the networks is empty. Customers, numbered $n = 1, 2, \ldots, N$, arrive at epochs $t(1), \ldots, t(N)$, respectively. The n-th customer of the input is sent to station ν_n^0 (it leaves the network immediately if $\nu_n^0 = K+1$) and is put at the end of the queue on this station.

As explained in the introduction, the queueing networks we consider here are characterized by the fact that service times and the switching sequences are associated with the stations. This means that the j-th service on station k ($j = 1, 2, \ldots; k = 1, \ldots, N$) takes σ_j^k units of time. In addition, when this service is completed, the leaving customer is immediately sent to station ν_j^k (it leaves the network if $\nu_j^k = N+1$) and it is put at the end of the queue on this station.

Remark 4 *Since we will only be interested in queue length processes and in view of our assumptions on the way services are allocated, we could replace FCFS by any non-preemptive, work-conserving discipline.*

3.2 First-Order State Variables

Let Ψ_j^k be the epoch at which the j-th service is completed on station k. By convention, we will take $\Psi_j^k = \infty$ if the j-th service time is never completed (remember that we consider networks with finite inputs, so that in the absence of 'capture' of customers, the total number of service completions on station k during the time interval $[t(1), \infty)$, which will be denoted Φ^k, is finite for all k). The following theorem shows that one can recursively construct the sequences Ψ_j^k, $k = 1, \ldots, K$, $j \geq 1$ from the above data (see Baccelli, Cohen and Gaujal 92 for the proof):

Theorem 4 *For $l = 0, \ldots, K$ and $k = 1, \ldots, K$, let $\eta^{l,k} : I\!N \to I\!N$ be the mapping*

$$\eta^{l,k}(j) = \inf\{m \geq 1 : \sum_{p=1}^{m} I(\nu_p^l = k) = j\}, \quad j \geq 1. \tag{13}$$

Define

$$\Psi_j^0 = \begin{cases} t(j) & \text{for } 1 \leq j \leq N; \\ \infty & \text{for } j > N. \end{cases} \tag{14}$$

Then the variables Ψ_j^k can be recursively computed from the following set of evolution equations

$$\Psi_j^k = \sigma_j^k + \max\left(\Psi_{j-1}^k, \min_{(j_0,j_1,\ldots,j_K)\in I\!N: j_0+j_1+\cdots+j_K=j} \left(\max_{l=0,\ldots,K} \Psi^l_{\eta^{l,k}(j_l)}\right)\right), \quad k = 1, \ldots, K, j \geq 1, \tag{15}$$

with initial conditions $\Psi_0^k = -\infty$, for $k = 1, \ldots, K$.

All other variables of interest to us can be obtained from these state variables:

- $\Psi_j^{k,l}$, $k = 0, 1, \ldots, K$, $l = 1, \ldots, K, K+1$, $j \geq 1$ will denote the sequence of service completion times on station k (arrivals of the input process for $k = 0$) such that the customer is sent to station l (leaves the network for $l = N + 1$);
- $\Phi^{k,l}$ will denote the total number of such epochs in $[t(1), \infty)$. Thus if $\Phi^{k,l} < \infty$, then $\Psi_j^{k,l} = \infty$ for $j > \Phi^{k,l}$. It is clear that $\Phi^k = \Phi^{k,1} + \cdots + \Phi^{k,K+1}$ for each k. In particular, $\Phi^{0,k} = \sum_{p=1}^{N} I(\nu_p^0 = k)$ will denote the total number of arrival epochs on station k (note that $N = \sum_k \Phi^{0,k}$).

Queue-length and service processes are completely defined by the sequences $\{\Psi_j^{k,l}\}$ (see below).

Remark 5 *Assume that $t(n-1) < t(n) = t(n+r) < t(n+r+1)$ for some $n \geq 1$, $r \geq 1$, $n + r \leq N$ (by convention, $t(0) = -\infty$, $t(N+1) = \infty$). If we replace the sequence $\{\nu_j^k\}$ by another one $\{\tilde{\nu}_j^k\}$, where*

- $\tilde{\nu}_j^k = \nu_j^k$ *for $k = 1, \ldots, K$, $j = 1, 2, \ldots$;*

- $\tilde{\nu}_j^0 = \nu_j^0$ *for* $j < m$ *and for* $j > m + r$;
- $\{\tilde{\nu}_j^0,\ m \leq j \leq m + r\}$ *is an arbitrary permutation of* $\{\nu_j^0; m \leq j \leq m + r\}$.

then the sequences $\{\Psi_j^{k,l}\}$ *do not change (the same is true in particular for queue-length and service processes).*

Remark 6 *If we replace the arrival epochs* $\{t(n)\}$ *by* $\{\hat{t}(n) \equiv t(n) + x\}$ *for some fixed* x *then the corresponding epochs* $\hat{\Psi}_j^k$ *and* $\hat{\Psi}_j^{k,l}$ *satisfy the equations* $\hat{\Psi}_j^k = \Psi_j^k + x; \hat{\Psi}_j^{k,l} = \Psi_j^{k,l} + x$ *for all* j, k, l.

Remark 7 *We will also consider the case of* delayed networks. *A delayed network is a network to which an extra sequence of real numbers* $\{\alpha_j^k\}$, $k = 1, \ldots, K$, $j \geq 1$ *is added. The rule is that the* j*-th service in station* k *cannot start before time* α_j^k. *The state variables* $\check{\Psi}_j^k$ *of the network* Σ *delayed with* α *are defined as follows:*

$$\check{\Psi}_j^k = \sigma_j^k + \max\left(\alpha_j^k, \check{\Psi}_{j-1}^k, \min_{(j_0,j_1,\ldots,j_K)\in\mathbb{N}: j_0+j_1+\cdots+j_K=j} \left(\max_{l=0,\ldots,K} \check{\Psi}^l_{\eta^{l,k}(j_l)}\right)\right). \tag{16}$$

3.3 Simple Networks

A simple network is a network with one external arrival only. Fix a successful route $r(1)$ with length $\varphi(1)$ and define the sequences $\nu^k(1)$ as in §2.1. Let $t(1)$ be some real number and $\{\sigma_j^k(1), 1 \leq j \leq \varphi^k(1)\}$ be an arbitrary sequence of non-negative real numbers.

Consider an open Jackson-type queueing network with K single-server stations and with only one customer arriving at epoch $t(1)$. This customer follows route $r(1)$. This means that it is first sent to station $r_1(1)$, receives a service there, is then sent to station $r_2(1)$ and so on. Finally (after the $\varphi(1)(1)$-th service) it leaves the network. As explained before, the j-th service on station k, $k = 1, \ldots, n$, $j = 1, \ldots, \varphi^k(1)$, takes $\sigma_j^k(1)$ units of time. Similarly, after this service, the customer is sent to station $\nu_j^k(1)$ (leaves the network if $\nu_j^k(1) = N + 1$), where $\nu_j^k(1)$ is the switching sequence associated with route $r(1)$ as defined in §2.1.

It is clear that such a simple network is both a particular case of the class of networks with finite input defined in §3.1 (we can take $\sigma_j^k = \sigma_j^k(1)$, $\Phi^k = \varphi^k(1)$ and $\nu_j^k = \nu_j^k(1)$, for $j \leq \varphi^k(1)$ and continue σ_j^k and ν_j^k in an arbitrary way for $j > \varphi^k(1)$) and a particular case of a Kelly-type network (since it is defined through the predefined route $r(1)$ of the customer and through the successive services that the customer will receive along this route).

3.4 Composition of Simple Networks

Let N be a positive integer, $R = ((r(1), \ldots, r(N))$ be a sequence of successful routes (see § 2.2). For each $n = 1, \ldots, N$, consider the simple network Σ_n with arrival epoch $t(n)$ and service times $\{\sigma_j^k(n), 1 \leq j \leq \varphi^k(n),\ k = 1, \ldots, K\}$. Assume that $t(1) \leq \ldots \leq t(N)$. Let $\nu^k[N]$ be defined as in § 2.2 and let

$$\begin{aligned}
T &= (t(1), \ldots, t(N)), \quad \varphi_N = \varphi(1) + \cdots + \varphi(N), \\
\varphi_N^k &= \varphi^k(1) + \cdots + \varphi^k(N), \quad \varphi_N^{k,l} = \varphi^{k,l}(1) + \cdots + \varphi^{k,l}(N), \\
\{\sigma_j^k[N]\}_{j=1}^{\varphi_N^k} &= \{\sigma_1^k(1), \ldots, \sigma_{\varphi^k(1)}^k(1), \ldots, \sigma_1^k(N), \ldots, \sigma_{\varphi^k(N)}^k(N)\}.
\end{aligned}$$

Let $\{\sigma_j^k\}_{j=1}^{\infty}$ and $\{\nu_j^k\}_{j=1}^{\infty}$ be two arbitrary sequences such that

$$\sigma_j^k = \sigma_j^k[N],\ \nu_j^k = \nu_j^k[N], \quad 1 \leq j \leq \varphi_N^k,\ k = 1, \ldots, N,$$

and

$$\nu_n^0 = r_1(n), \quad n = 1, \ldots, N.$$

Definition 5 *A queueing network $\Sigma = (N, T, \sigma, \nu)$ with finite input T is a composition of N simple networks with respective routes $r(1), \ldots, r(N)$ if it admits the above representation. We shall then write $\Sigma = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_N$, where the $+$ operation is clearly associative.*

Remark 8 *If we have an infinite sequence of simple networks, say Σ_n, $n \geq 1$, we can also consider the network $\Sigma[\infty] = \Sigma_1 + \Sigma_2 + \cdots$. Let $\Sigma[N] = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_N$. It is easily checked that if $\Sigma[\infty] = (\infty, T, \sigma, \nu)$, then the queueing process (see below) in $\Sigma[N]$ coincides with that of $\Sigma'[\infty] = (\infty, T', \sigma, \nu)$, where $T' = (t(1), t(2), \ldots, t(N), \infty, \infty, \ldots)$.*

The most important result for such networks is that the number of service completions on each queue does not depend on the sequence T:

Theorem 5 (Conservation rule) *If a network is a composition of N simple networks then*

$$\Phi = \varphi_N,\ \Phi^k = \varphi_N^k,\ \Phi^{k,l} = \varphi_N^{k,l} \quad \forall k, l, \tag{17}$$

regardless of T and σ.

Proof It is sufficient to show that $\Phi = \varphi_N$. But this is a direct corollary of Theorem 2 (see also Appendix 8.2 for another simple proof). ∎

3.5 Decomposition of Networks

Consider a queueing network $\Sigma = (N, T, \sigma, \nu)$. The following result holds.

Theorem 6 *If $\Phi^k < \infty$ for all $k = 1, \ldots, K$ then one can construct N simple networks Σ_n, $n = 1, \ldots, N$, with some routes $r(1), \ldots r(N)$, such that Σ is the composition of these simple networks.*

Proof This is a direct corollary of Theorem 3. ∎

3.6 Composition of Networks

Consider two non necessarily simple networks $\Sigma_0 = (N_0, T_0, \sigma_0, \nu_0)$ and $\Sigma_1 = (N_1, T_1, \sigma_1, \nu_1)$, where $t_0(N_0) \leq t_1(1)$ and where the switching sequences $\{\nu^k_{j,0}\}$ are such that Φ_0 is finite.

By definition, the composition of Σ_0 and Σ_1 is the network $\Sigma = (N, T, \sigma, \nu)$ defined by the following relations: $N = N_0 + N_1$,

$$T = (t_0(1), \ldots, t_0(N_0), t_1(1), \ldots, t_1(N_1)), \tag{18}$$

and

$$\sigma^k_j = \begin{cases} \sigma^k_{j,0}, & \text{for } 1 \leq j \leq \Phi^k_0; \\ \sigma^k_{j-\Phi^k_0,1}, & \text{for } j > \Phi^k_0 \end{cases}$$

and

$$\nu^k_j = \begin{cases} \nu^k_{j,0} & \text{for } 1 \leq j \leq \Phi^k_0; \\ \nu^k_{j-\Phi^k_0,1} & \text{for } j > \Phi^k_0. \end{cases}$$

In view of the decomposition property of Theorem 6 and of the associativity property of +, it makes sense to use the notation $\Sigma = \Sigma_1 + \Sigma_2$.

3.7 Monotonicity and Continuity Properties

For fixed K, N, ν and σ consider now two different input sequences: $T = \{t(n)\}^N_{n=1}$ and $\tilde{T} = \{\tilde{t}(n)\}^N_{n=1}\}$, and the two queueing networks: $\Sigma = (N, , T, \sigma, \nu)$ and $\tilde{\Sigma} = (N, , \tilde{T}, \sigma, \nu)$. The main monotonicity property is:

Theorem 7 *If $t(n) \le \tilde{t}(n)$ for each $n = 1, \ldots, N$, then $\Psi_j^k \le \tilde{\Psi}_j^k$ and $\Psi_j^{k,l} \le \tilde{\Psi}_j^{k,l}$* for all j, k, l.

Proof The first proofs of this result are that of [17] and [30]. The proof and some extensions of this results which will be needed later on also follows from an induction argument based on the evolution equations of Theorem 4 (see [4]). ∎

We now show a couple of corollaries of this result.

Corollary 1 *If $t(n) \le \tilde{t}(n) \le t(n) + x$ for all $n = 1, \ldots, N$, and for some $x > 0$, then*

$$\Psi_j^k \le \tilde{\Psi}_j^k \le \Psi_j^k + x \tag{19}$$

and

$$\Psi_j^{k,l} \le \tilde{\Psi}_j^{k,l} \le \Psi_j^{k,l} + x \tag{20}$$

for all j, k, l.

Proof Introduce a new network $\hat{\Sigma} = (N, \{t(n) + x\}_{n=1}^N, \sigma, \nu)$. It follows from Theorem 7 that $\Psi_j^k \le \tilde{\Psi}_j^k \le \hat{\Psi}_j^k$ and from Remark 6 that $\hat{\Psi}_j^k = \Psi_j^k + x$ (the same holds for $\{\Psi_j^k\}$). ∎

Corollary 2 *Consider two networks: $\Sigma = (N, T, \sigma, \nu)$ and $\hat{\Sigma} = (N, T, \hat{\sigma}, \nu)$ with the same input and switching sequences but with different service times. If $\hat{\sigma}_{j_0}^{k_0} = \sigma_{j_0}^{k_0} + x$ for some $k_0 \in \{1, \ldots, K\}$ and $x > 0$, and $\hat{\sigma}_j^k = \sigma_j^k$ for all $(j, k) \ne (j_0, k_0)$, then*

$$\Psi_j^k \le \hat{\Psi}_j^k \le \Psi_j^k + x, \quad \forall j, k$$

(the same property holds for $\{\Psi_j^{k,l}\}$).

Proof The proof is similar to that of Corollary 1. Another simple proof can obtained by an induction based on the equations of Theorem 4. ∎

Consider now two networks $\Sigma = (N, T, \sigma, \nu)$ and $\tilde{\Sigma} = (N, \tilde{T}, \tilde{\sigma}, \nu)$ with the same switching sequences. Assume in addition that $\Phi^k < \infty$ for all k.

Corollary 3 *If $t(n) \le \tilde{t}(n)$ for all $n = 1, \ldots, N$ and $\sigma_j^k \le \tilde{\sigma}_j^k$ for all $k = 1, \ldots, K$, $j = 1, \ldots, \Phi^k$, then*

$$\Psi_j^k \le \tilde{\Psi}_j^k \le \Psi_j^k + \max_{1 \le n \le N}(\tilde{t}(n) - t(n)) + \sum_{l=1}^{K} \sum_{i=1}^{\Phi^l} (\tilde{\sigma}_i^l - \sigma_i^l) \tag{21}$$

for all j, k (the same holds true for $\Psi_j^{k,l}$).

Proof This result follows immediately from Corollaries 1-2 and from induction arguments. ∎

Remark 9 *(Continuation of Remark 7) It is easy to check that if $\alpha_j^k(1) \le \alpha_j^k(2)$ for all j and k, then the network Σ, when delayed with $\alpha(1)$ and $\alpha(2)$ respectively, leads to state variables that satisfy the relation*

$$\check{\Psi}_j^k(1) \le \check{\Psi}_j^k(2), \quad \forall j, k.$$

In particular, a delayed network is always a majorant of the non-delayed network in the sense mentioned above.

Fix now $K, N, \{\nu_j^k\}$ and consider a set of sequences $\{t_\epsilon(n)\}_{n=1}^N$ and $\{\sigma_{j,\epsilon}^k\}_{j=1}^\infty$, for $k = 1, \ldots, K$, where $\epsilon > 0$.

Corollary 4 (Continuity property) *Assume that Φ is finite and that*

$$t_\epsilon(n) \to t(n), \ \sigma_{j,\epsilon}^k \to \sigma_j^k \tag{22}$$

as $\epsilon \to 0$ for all $n = 1, \ldots, N$, $k = 1, \ldots, K$, $j = 1, \ldots, \Phi^k$. Then

$$\Psi_{j,\epsilon}^k \to \Psi_j^k \tag{23}$$

for each $k = 1, \ldots, K$, $j = 1, \ldots, \Phi^k$ (the same holds true for $\Psi_{j,\epsilon}^{k,l}$).

Proof The proof follows immediately from Corollary 3. ∎

Corollary 5 *Let Σ be the composition of the networks Σ_0 and Σ_1. Then*

$$\Psi_{j+\Phi_0^k}^k \ge \Psi_{j,1}^k \tag{24}$$

and

$$\Psi_i^k \le \Psi_{i,0}^k \tag{25}$$

for each $k = 1, \ldots, K$, $j = 1, 2, \ldots,$ $i = 1, \ldots, \Phi^k$ (the same holds for $\Psi_j^{k,l}$).

Proof We prove (24) only (the proof of (25) is similar). We construct an auxiliary network $\tilde{\Sigma}_2$ with driving sequences $(\{\tilde{t}_2(n)\}_{n=1}^{N_2}, \{\tilde{\sigma}_{j,2}^k\}, \{\tilde{\nu}_{j,2}^k\})$ obtained by shifting the sequence t_2 to the left in such a way that the two networks Σ_0 and Σ_1 *separate*, namely the last departure from the customers of the first network takes place before the first arrival of the second network of the composition. More precisely, let

$$\Delta = \max_{0 \le l \le N_0} \{\Psi^l_{\Phi_0^l,0} - t_0(1)\}. \tag{26}$$

We take

- $\tilde{\sigma}_{j,2}^k = \sigma_{j,2}^k$ and $\tilde{\nu}_{j,2}^k = \nu_{j,2}^k$ for all j,k;
- $\tilde{t}_2(1) = \min\{t_0(1), t_1(1) - \Delta\}$, $\tilde{t}_2(n+1) = \tilde{t}_2(n) + t_0(n+1) - t_0(n)$ for $n \le N_0$ and $\tilde{t}_2(n) = t_2(n)$ for $n > N_0$.

Since $\tilde{t}_2(n) \le t_2(n)$ for all $n \ge 1$, Theorem 7 implies that

$$\tilde{\Psi}_{j,2}^k \le \Psi_{j,2}^k \tag{27}$$

for all j,k. But since the last customer of $\tilde{\Sigma}_0$ leaves the network before the arrival of the first customer of Σ_1 (it is in that sense that the networks are separated), then

$$\tilde{\Psi}^k_{j+\Phi_0^k,2} = \Psi_{j,1}^k$$

for all j,k. ∎

Remark 10 *The notion of separation of the composition of two networks which is introduced in the proof of the preceding corollary is quite crucial and will be used at several occasions later on.*

3.8 The Space D_+^0

Let $f : [0,\infty) \to \{0,1,2,\ldots\}$ be a right-continuous non-increasing function with compact support, i.e.

$$\beta_f \equiv \sup\{x : f(x) > 0\} < \infty \tag{28}$$

and $D_+^0 \equiv D_+^0[0,\infty)$ be the space of such functions.

Let H be the set of all continuous, strictly increasing functions $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, $h(\infty) = \infty$. For $f, g \in D_+^0$, consider the metric

$$d(f,g) = \inf_{h\in H}\{\sup_{x\ge 0}|h(x) - x| + \sup_{x>0}|f(h(x)) - g(x)|\}. \tag{29}$$

For $f \in D_+^0$, set $a(f) = f(0)$ and $b(f) = \inf\{x \geq 0 : f(x) = 0\}$.

Note that the space (D_+^0, d) is separable and possesses the following properties:

- It admits the partial order $\leq$ defined by $f \leq g$ if $f(x) \leq g(x)$ for all $x \geq 0$.
- If the sequence $\{f_n\}, f_n \in D_+^0$ is Cauchy (w.r. to d), then there exists a function $g \in D_+^0$ such that $g \geq f_n$, for all $n \geq 0$.
- If the sequence $\{f_n\}, f_n \in D_+^0$ is monotone increasing (non-decreasing) and if $\lim_n a(f_n) < \infty$ and $\lim_n b(f_n) < \infty$, then $\lim_n f_n \equiv f \in D_+^0$.

3.9 Second-Order State Variables

Fix $N \geq 1$ and consider a network with N customers. Assume Φ to be finite. For each k, l consider the processes

$$\bar{\Gamma}^{k,l}(t) \quad = \quad \Phi^{k,l} - \sum_{j=1}^{\Phi^{k,l}} I(\Psi_j^{k,l} \leq t) \tag{30}$$

$$\bar{\Gamma}^{k}(t) \quad \equiv \quad \sum_{l=1}^{K+1} \bar{\Gamma}^{k,l}(t) = \varphi^k - \sum_{j=1}^{\varphi^k} I(\Psi_j^{k} \leq t), \tag{31}$$

(where $\varphi^0 = N$) and $\Psi_j^0 = t(j)$, which count the number of departures from station k to station l (resp. from station l) taking place after time t.

The processes $\bar{\Gamma}^{k,l}(t)$ and $\bar{\Gamma}^{k}(t)$ are right-continuous and belong to D_+^0.

We will also need the following processes:

- $\bar{Q}^k(t)$ is the queue-length on station k at time t (including the customer in service);
- $\bar{\chi}^k(t)$ is the residual service time of the customer in service at time $t+$ in station k (0 if $\bar{Q}^k(t) = 0$).

These processes are defined from the $\bar{\Gamma}$ functions through the following relations:

$$\bar{Q}^k(t) \quad = \quad \bar{\Gamma}^k(t) - \sum_{l=0}^{K} \bar{\Gamma}^{l,k}(t); \tag{32}$$

$$\begin{aligned}\bar{Q}(t) &\equiv \sum_{k=1}^{K}\bar{Q}^k(t) = \sum_{k=1}^{K}\bar{\Gamma}^k(t) - \sum_{k=1}^{K}\sum_{l=0}^{K}\bar{\Gamma}^{l,k}(t)\\ &= \sum_{k=1}^{K}\bar{\Gamma}^{k,K+1}(t) - \sum_{k=1}^{K}\bar{\Gamma}^{0,k}(t); \qquad (33)\\ \bar{\chi}^k(t) &= \inf\{v > t : \bar{\Gamma}^k(v) < \bar{\Gamma}^k(t)\} - t; \qquad (34)\end{aligned}$$

where the last relation assumes that $\bar{Q}^k(t) > 0$.

¿From Theorem 7 and its corollaries, we get (see also [17], [18]):

Lemma 1 *Consider two networks:* $\Sigma_1 = (N, T_1, \sigma_1, \nu_1\})$ *and* $\Sigma_2 = (N, T_2, \sigma_2, \nu_2\})$. *If* $t_1(n) \leq t_2(n)$, $\sigma^k_{j,1} \leq \sigma^k_{j,2}$ *and* $\nu^k_{j,1} = \nu^k_{j,2}$, *for all* $n = 1, \dots, N$, $k = 1, \dots, K$, $j = 1, 2, \dots$, *then*

$$\bar{\Gamma}_1^{k,l}(t) \leq \bar{\Gamma}_2^{k,l}(t) \qquad (35)$$

for all k, l *and for all* $-\infty < t < \infty$.

We will also need the functions describing the residual departure processes and the residual queue length processes. Let

$$\Gamma^{k,l}(t) = \bar{\Gamma}^{k,l}(t + t(N)), \quad t \geq 0 \qquad (36)$$

and

$$\Gamma^k(t) = \bar{\Gamma}^k(t + t(N)), \quad t \geq 0. \qquad (37)$$

Note that

$$\Gamma^k(t) = \sum_{l=1}^{K+1}\Gamma^{k,l}(t), \quad t \geq 0. \qquad (38)$$

Remark 11 *The processes* $\Gamma^{k,l}(t)$ *do not depend on the values* $t(1), \dots, t(N)$ *but only on their increments* $t(n+1) - t(n)$, $n = 1, \dots, N-1$. *This means, in particular, that if we consider two networks* Σ *and* $\hat{\Sigma}$ *with the same service times and switching decisions and with inputs* $\{t(n)\}$ *and* $\{\hat{t}(n)\}$ *satisfying the equations* $\hat{t}(n) = t(n) + C$ *for some* $C \geq 0$ *and for all* n, *then* $\Gamma^{k,l}(t) = \hat{\Gamma}^{k,l}(t)$ *for all* t, k, l *(the same is true for* $\Gamma^k(t)$*).*

Let $\tau(n) = t(n+1) - t(n), n = 1, \dots, N-1$.

Lemma 2 **(Monotonicity property)** *Consider two networks* $\Sigma_1 = (N, T_1, \sigma_1, \nu_1)$ *and* $\Sigma_2 = (N, T_2, \sigma_2, \nu_2)$. *If* $\tau_1(n) \geq \tau_2(n)$, $\sigma^k_{j,1} \leq \sigma^k_{j,2}$ *and* $\nu^k_{j,1} = \nu^k_{j,2}$ *for all* $n = 1, \ldots, N-1$, $k = 1, \ldots, K$, $j = 1, 2, \ldots$, *then*

$$\Gamma^{k,l}_1(t) \leq \Gamma^{k,l}_2(t) \tag{39}$$

for all k, l, t.

Proof The processes are compared at different epochs: $t + t_1(N)$ and $t + t_2(N)$, respectively. For connecting these two epochs, introduce two new networks:

$$\tilde{\Sigma}_1 = (N, \{\tilde{t}_1(n)\}^N_{n=1}, \{\sigma^k_{j,1}\}, \{\nu^k_{j,2}\}) \tag{40}$$

and

$$\tilde{\Sigma}_2 = (N, \{\tilde{t}_2(n)\}^N_{n=1}, \{\sigma^k_{j,2}\}, \{\nu^k_{j,2}\}) \tag{41}$$

where $\tilde{t}_1(n) = C - \sum_{j=n}^{N-1} \tau_1(j)$, for $n < N$, $\tilde{t}_1(N) = C$, $\tilde{t}_2(n) = C - \sum_{j=n}^{N-1} \tau_2(j)$, for $n < N$, $\tilde{t}_2(N) = C$ and $C = \max(t_1(N), t_2(N))$.

¿From Remark 11,

$$\tilde{\Gamma}^{k,l}_1(t) = \Gamma^{k,l}_1(t) \tag{42}$$

and

$$\tilde{\Gamma}^{k,l}_2(t) = \Gamma^{k,l}_2(t) \tag{43}$$

for all k, l, t. Since $\tilde{t}_1(n) \leq \tilde{t}_2(n)$ for each n, then $\tilde{\Gamma}^{k,l}_1(t) \leq \tilde{\Gamma}^{k,l}_2(t)$ for all k, l, t. ■

Similarly, the residual queue-length processes and the residual service-time processes are defined by the relations:

$$Q^k(t) = \bar{Q}^k(t + t(N)), \quad Q(t) = \bar{Q}(t + t(N)), \tag{44}$$

$$\chi^k(t) = \bar{\chi}^k(t + t(N)), \quad t \geq 0. \tag{45}$$

We have

$$Q^k(t) = \Gamma^k(t) - \sum_{l=1}^{K} \Gamma^{l,k}(t) \equiv \sum_{i=1}^{K+1} \Gamma^{k,i}(t) - \sum_{l=1}^{K} \Gamma^{l,k}(t), \tag{46}$$

$$\chi^k(t) = \inf\{v > t : \Gamma^k(v) < \Gamma^k(t)\} - t \tag{47}$$

if $Q^k(t) > 0$ ($\chi^k(t) = 0$ if $Q^k(t) = 0$), and

$$Q(t) = \sum_{k=1}^{K} \Gamma^{k,K+1}(t) \tag{48}$$

for $k = 1, \ldots, K$, $t \geq 0$.

Corollary 6 *Under the conditions of Lemma 2,*

$$Q_1(t) \leq Q_2(t) \tag{49}$$

for all $t \geq 0$.

Returning now to the composition of networks (see § 3.6), we can formulate the following immediate corollary of Lemma 2:

Corollary 7 *If the network* Σ *is the composition of two networks* Σ_0 *and* Σ_1, *and if* Φ_0 *and* Φ_1 *are finite, then*

$$\Gamma^{k,l}(t) \geq \Gamma_1^{k,l}(t), \quad Q(t) \geq Q_1(t) \tag{50}$$

for all k, l, t.

Associated with any network Σ, we introduce the new variable:

$$Z = \inf\{t \geq 0 : \max_{1 \leq k \leq K} \Gamma^k(t) = 0\}, \tag{51}$$

which represents the time to empty the system, measured from the last external arrival.

Lemma 3 *If* Σ *is the composition of the networks* Σ_0 *and* Σ_1, *then*

$$(Z - x - y)^+ \leq (Z_0 - x)^+ + (Z_1 - y)^+, \tag{52}$$

for all $x \geq 0$, $y \geq 0$.

Proof It is enough to consider the case $x = y = 0$ only. If $z \equiv t_1(1) - t_0(N_0) \geq Z_0$, then the two networks are separated and $Z = Z_1 \leq Z_0 + Z_1$. If $z < Z_0$, let $\hat{\Sigma}_2$ be the composition of the networks Σ_0 and $\hat{\Sigma}_1$, where

$$\hat{\Sigma}_1 = (\{\hat{t}_1(n)\}_{n=1}^{N_1}, \{\sigma_{j,1}^k\}, \{\nu_{j,1}^k\})$$

with $\hat{t}_1(n) = t_1(n) + (Z_0 - z)$, $n = 1, \ldots, N_1$. By construction, $\hat{Z}_2 = Z_1$. Lemma 1 implies that $Z_2 + t_1(N_1) \leq \hat{Z}_2 + t_1(N_1) + Z_0 - z$. So $Z_2 \leq Z_1 + Z_0$ (z is non-negative by definition). ∎

Remark 12 *The same monotonicity and sub-additive properties hold true for networks with multi-server stations (with FCFS disciplines), provided we still associate service times and switching decisions with stations. More precisely, we have to assume that, on each station* k, *the* j*-th service takes* σ_j^k *units of time (regardless of the server to which the customer is allocated), and that after this service, the customer is sent to station* ν_j^k *(see [30] for the monotonicity property).*

4 First-Order Ergodic Properties

4.1 Basic Definitions and Notations

Consider a sequence of simple networks say $\{\Sigma(n)\}_{n=-\infty}^{\infty}$, where $\Sigma(n) = (1, t(n), \sigma(n), \nu(n))$ and where the switching decisions $\nu(n)$ is that generated by the route $r(n) = (r_1(n), r_2(n), \ldots)$. We assume that $t(n) \le t(n+1)$ for all n and we denote $\tau(n)$ the difference $t(n+1) - t(n)$. Associated with the sequence $\{\Sigma(n)\}$, we define the following basic sequences $u(n)$, $\{S^k(n)\}$, $\{F^k(n)\}$, $\{s_j^k\}$ and $\{n_j^k\}$:

- $u(0) = 0$ and $u(n+1) - u(n) = \tau(n)$ for all n;
- $S^k(n) = \sum_{j=1}^{\varphi^k(n)} \sigma_j^k(n)$ and $S(n) = S^1(n) + \cdots + S^K(n)$ for all $-\infty < n < \infty$, $k = 1, \ldots, K$;
- $F^k(n) = \varphi^k(1) + \cdots + \varphi^k(n)$ for $n \ge 1$ and $F^k(n) = \varphi^k(n) + \cdots + \varphi^k(0)$ for $n \le 0$;
- $\nu_j^0 = r_1(j)$, for $-\infty < j < \infty$;
- for $k = 1, \ldots, K$,
  - for $0 < j \le \varphi^k(1)$ $s_j^k = \sigma_j^k(1)$ and $n_j^k = \nu_j^k(1)$;
  - for $n \ge 1$, $F^k(n) < j \le F^k(n+1)$ $s_j^k = \sigma_{j-F^k(n)}^k(n+1)$ and $n_j^k = \nu_{j-F^k(n)}^k(n+1)$;
  - for $-\varphi^k(0) < j \le 0$ $s_j^k = \sigma_{j+\varphi^k(0)}^k(0)$ and $n_j^k = \nu_{j+\varphi^k(0)}^k(0)$;
  - for $n \ge 0$, $-F^k(-n-1) < j \le -F^k(-n)$, $s_j^k = \sigma_{j+F_{-n-1}^k}^k(-n-1)$ and $n_j^k = \nu_{j+F_{-n-1}^k}^k(-n-1)$.

Assume that we have a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, endowed with an ergodic measure-preserving shift θ. The symbols $\theta^n, n \ge 0$, will denote the iterations of this transformation (so that $\theta^1 = \theta$, while θ^0 is the identity), and the symbol θ^{-n} stands for the transformation inverse to θ^n, $n = 1, 2, \ldots$. The same symbol θ will also be used for the measure-preserving shift on the events of $\mathcal{F}$. Let

$$\xi(n) = \{\tau(n), \{\sigma_j^k(n)\}, \{\nu_j^k(n)\}\}. \tag{53}$$

Our stochastic assumptions will be as follows:

- all the variables $t(n)$, $\{\sigma_j^k(n)\}$, $\{\nu_j^k(n)\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$;
- the random variables $\xi(n)$ satisfy the relation $\xi(n) = \xi(0) \circ \theta^n$ for all n, which implies that $\{\xi(n)\}_{n=-\infty}^{\infty}$ is stationary and ergodic;

- all the expectations $\mathbf{E}\varphi(0)$, $\mathbf{E}S^k(0) = b^k$, $\mathbf{E}\tau(0) = \lambda^{-1}$ are finite.

Without loss of generality, we can assume that $\mathbf{E}S^k(0) > 0$ for all k.

For $m \leq n$ let

$$\Sigma_{[m,n]} = \Sigma(m) + \cdots + \Sigma(n),$$

where $+$ is the composition rule introduced in § 3.4. We have in particular $\Sigma_{[n,n]} = \Sigma(n)$. The composition and decomposition results of § 3.4-3.5 imply that for each $m < l \leq n$,

$$\Sigma_{[m,n]} = \Sigma_{[m,l-1]} + \Sigma_{[l,n]}.$$

Let $X_{[m,n]}$ be the time to empty the system measured from time $t(0)$:

$$X_{[m,n]} = t(n) - t(0) + Z_{[m,n]}, \tag{54}$$

where $Z_{[m,n]}$ represents the variable defined in (51) for the network $\Sigma_{[m,n]}$, for $-\infty < m \leq n < \infty$. We shall also use the notation

$$X_n = X_{[0,n]} - t(0). \tag{55}$$

4.2 First-Order Ergodic Theorem

The variable X_n, which can be seen as the maximum over all j and k of the Ψ_j^k variables in network $\Sigma_{[0,n]}$ measured from t_0 (and equivalently the variables $Z_{[0,n]}$ or $Z_{[-n,0]}$) satisfy a SLLN:

Theorem 8 *Under the above conditions, there exists a finite non-negative constant γ such that*

$$\lim Z_{[-n,0]}/n = \lim Z_{[-n,-1]}/n = \lim \mathbf{E}Z_{[-n,0]}/n = \lim \mathbf{E}Z_{[-n,-1]}/n = \gamma \tag{56}$$

a.s. as $n \to \infty$.

Proof It follows from Lemma 3 that

$$Z_{[-n,-1]} \leq Z_{[-n,-l-1]} + Z_{[-l,-1]} \tag{57}$$

for all $1 \leq l < n$. Since $Z_{[-n,-l-1]} = Z_{[-n+l,-1]} \circ \theta^{-l}$ and $0 \leq \mathbf{E}Z_{[0,0]} \leq \mathbf{E}S(0) < \infty$, Kingman's subadditive ergodic theorem allows us to complete the proof. ∎

Corollary 8 *Under the above conditions*

$$\lim Z_{[1,n]}/n = \lim \mathbf{E} Z_{[1,n]}/n = \gamma \tag{58}$$

and

$$\lim X_n/n = \lim \mathbf{E} X_n/n = \gamma + \lambda^{-1} \tag{59}$$

a.s. as n tends to ∞.

Remark 13 *Consider the more general situation when the sequence $\{\xi(n)\}_{n=0}^{\infty}$ couples with a stationary sequence. If the stationary sequence under consideration satisfies the above assumptions, then*

$$\lim \frac{Z_{[1,n]}}{n} = \gamma, \quad \lim \frac{X_n}{n} = \gamma + \lambda^{-1} \tag{60}$$

a.s. as $n \to \infty$. If, in addition, all the expectations $\mathbf{E}\varphi(n)$, $\mathbf{E}S^k(n)$, $\mathbf{E}\tau(n)$ are finite and the coupling time is integrable, then the statement of Corollary 8 is still true.

4.3 Finiteness of Second-Order Variables

The monotonicity property (7) implies that $Z_{[-n-1,0]} \geq Z_{[-n,0]}$ a.s. for all non-negative n. So there exists an a.s. limit $\lim Z_{[-n,0]}$ as n tends to ∞ (which may be either finite or infinite).

In relation with the network $\Sigma_{[-n,0]}$ and for $k, 1, \ldots, K$, and $t \geq 0$, we also introduce the processes $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^k_{[-n,0]}(t)$, $\Gamma_{[-n,0]}(t)$, $Q^k_{[-n,0]}(t)$ and $Q_{[-n,0]}(t)$, which are defined as in § 3.9. Let

$$\Gamma^k_{[-n,0]} = \Gamma^k_{[-n,0]}(0); \tag{61}$$

and

$$D^k_{[-n,0]} = \sum_{j=0}^{\Gamma^k_{[-n,0]}-1} s^k_{-j}, \quad E^k_{[-n,0]} = \sum_{j=0}^{\Gamma^k_{[-n,0]}-2} s^k_{-j} \tag{62}$$

(here $\sum_0^{-1} = \sum_0^{-2} \equiv 0$). The monotonicity property also implies that $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^k_{[-n,0]}(t)$, $Q_{[-n,0]}(t)$, $\Gamma^k_{[-n,0]}$, $D^k_{[-n,0]}$ and $E^k_{[-n,0]}$ are non-decreasing in n. It follows from the definitions that

$$\max_{1 \leq k \leq K} E^k_{[-n,0]} \leq Z_{[-n,0]} \leq \sum_{k=1}^{K} D^k_{[-n,0]} \tag{63}$$

for all n. So $Z_{[-n,0]} \to \infty$ as $n \to \infty$ iff there exists a.s. $k \in \{1, \ldots, K\}$ such that $\Gamma^k_{[-n,0]} \to \infty$ as $n \to \infty$ (this is true because we assume $\mathbf{E}S^k(0)$ to be positive for all k).

Let A be the event

$$A = \{\lim_{n\to\infty} Z_{[-n,0]} = \infty\}. \qquad (64)$$

Theorem 9 *Under the conditions of § 4.1, either* $\mathbf{P}(A) = 1$ *or* $\mathbf{P}(A) = 0$.

Proof We shall prove that a.s. if $Z_{[-n,0]} \to \infty$ then $Z_{[-n,1]} \to \infty$ as n tends to ∞ . If it is so, then $\theta A \supseteq A$. But the shift θ is measure-preserving, so $\mathbf{P}(\theta A - A) = 0$. Since θ is ergodic, the last equality implies $\mathbf{P}(A) \in \{0,1\}$.

It follows from (63) that it is enough to prove that $\mathbf{P}(\Gamma^k_{[-n,0]} \to \infty) \in \{0,1\}$. For this it is sufficient to show that a.s. if $\Gamma^k_{[-n,0]} \to \infty$, then $\Gamma^k_{[-n,1]} \to \infty$.

For each $N >> 1$ we can a.s. choose $l \equiv l_N$ such that $\sum_{j=N+1}^{N+l} s^k_{-j} \geq \tau(0)$. Since $\Gamma^k_{[-n,0]} \to \infty$, there exists n_N such that $\Gamma^k_{[-n,0]} > N + l$ for all $n \geq n_N$. Therefore $\Gamma^k_{[-n,1]} \geq N$ for all $n \geq n_N$. ∎

Corollary 9 *If* $P(A) = 0$ *then the random variables* $Z_{[-n,0]}$ *converge monotonically a.s. to a finite random variable* $Z(0)$, *and if we define*

$$Z(m) = Z(0) \circ \theta^m$$

then

$$Z_{[-n+m,m]} \equiv Z_{[-n,0]} \circ \theta^m \leq Z(m) \quad a.s. \qquad (65)$$

for all $0 \leq m, n < \infty$.

It follows from Theorem 9 and from its proof that similar results hold true for the processes $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^k_{[-n,0]}(t)$, $\Gamma_{[-n,0]}(t)$ and $Q_{[-n,0]}(t)$:

Corollary 10 *If* $\mathbf{P}(A) = 0$ *then the processes* $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^k_{[-n,0]}(t)$, $\Gamma_{[-n,0]}(t)$ *and* $Q_{[-n,0]}(t)$ *converge monotonically a.s. to finite processes* $\Gamma^{k,l}(t) \in D_0^+$, $\Gamma^k(t) \in D_0^+$, $\Gamma(t) \in D_0^+$ *and* $Q(t) \in D_0^+$, *respectively.*

Denote by $Q^k_{[-n,0]} \equiv Q^k_{[-n,0]}(0)$ the queue-length and by $\chi^k_{[-n,0]} \equiv \chi^k_{[-n,0]}(0)$ the residual service time on station k in the network $\Sigma_{[-n,0]}$ at time $t(0)$ (where $\chi^k_{[-n,0]} = 0$ if $Q^k_{[-n,0]} = 0$).

Corollary 11 *If* $\mathbf{P}(A) = 0$, *then the r.v.'s* $Q^k_{[-n,0]}$ *and* $\chi^k_{[-n,0]}$ *converge weakly to some a.s. finite r.v.'s* Q^k *and* χ^k, *respectively, as* $n \to \infty$.

4.4 More on First-Order Ergodic Theorems

The results of this section will not be used in what follows. The main result is the solidarity property of Corollary 14.

For $j = -F^k(-n) + 1, \ldots, 0$ let

$$\psi^k_{j,[-n,0]} \tag{66}$$

be the epoch of the completion of the j-th service on station k in $\Sigma_{[-n,0]}$ and let

$$Z^k_{[-n,0]} = \psi^k_{0,[-n,0]} \tag{67}$$

be the moment of the completion of the last service on station k in $\Sigma_{[-n,0]}$ (with the convention that $= -\infty$ if $F^k_{[-n,0]} = 0$); similarly, for $l \leq n$, let

$$Z^k_{-l,[-n,0]} = \psi^k_{-F^k_{-l},[-n,0]} \tag{68}$$

$(= -\infty$ if $F^k(-n) = F^k(-l))$,

$$Z_{-l,[-n,0]} = \max_{1 \leq k \leq K} Z^k_{-l,[-n,0]} \tag{69}$$

and

$$S_{[m,n]} = \sum_{j=m}^{n} S(j). \tag{70}$$

Note that

$$Z_{[-n,0]} = \max_{1 \leq k \leq K} (Z^k_{[-n,0]})^+$$

and that

$$(Z_{-l,[-n,0]})^+ \leq Z_{[-n,0]}.$$

Lemma 4 *For all* $0 \leq l \leq n$

$$Z_{[-n,0]} \leq (Z_{-l,[-n,0]})^+ + S_{[-l,0]} \tag{71}$$

Proof The case $Z_{[-n,0]} = 0$ is trivial. Assume that $Z_{[-n,0]} > 0$. At each instant of the time interval $(0, Z_{[-n,0]})$ at least one customer is being served. In addition, from time $(Z_{-l,[-n,0]})^+$ on, the services completed on station k have an index larger than $(-F^k(-l)+1)$, for all $k = 1, \ldots, K$. Since $(Z_{-l,[-n,0]})^+ \geq 0$, $Z_{[-n,0]} - (Z_{-l,[-n,0]})^+$ is bounded from above by $S_{[-l,0]}$. ∎

Corollary 12 *If* $\mathbf{P}(A) = 1$ *then, for all fixed* $l \leq n$, $Z_{-l,[-n,0]} \to \infty$ *a.s. as* n *tends to* ∞.

Proof Let $\beta = \min\{n \geq 0 : Z_{-l,[-n,0]} \geq 0\}$. Then $(Z_{[-n,0]} - Z_{-l,[-n,0]})I(n \geq \beta) \leq S_{[-l,0]}$ a.s.. In particular,

$$(Z_{[-n,0]} - Z_{-l,[-n,0]})/g(n) \to 0 \tag{72}$$

a.s. as $n \to \infty$ for any function $g(n)$ such that $\lim g(n) = \infty$. ∎

Corollary 13 *Let* $l = l(n)$ *be some non-decreasing (possibly random) integer-valued function of* n *such that* $l(n)/g(n) \to 0$ *(a.s.) as* $n \to \infty$, *for* g *as above. Then*

$$(Z_{[-n,0]} - Z_{-l(n),[-n,0]})I(Z_{-l(n),[-n,0]} \geq 0)I(l(n) \leq n)/g(n) \to 0 \tag{73}$$

a.s. as n *tends to* ∞. *In particular, if* $l(n) = l$ *with* l *random, then* $\mathbf{P}(A) = 1$ *implies that* $Z_{-l,[-n,0]} \to \infty$ *a.s. and in addition,*

$$\limsup(Z_{[-n,0]} - Z_{-l,[-n,0]}) \leq S_{[-l,0]} \tag{74}$$

a.s. as n *tends to* ∞.

Proof If $l(\infty) \equiv \lim_n l(n)$ is finite, then the result follows from Corollary 12 and from monotonicity properties. If $l(\infty) = \infty$, then write

$$\frac{S_{[-l(n),0]}}{g(n)} = \frac{S_{[-l(n),0]}/l(n)}{g(n)/l(n)} \tag{75}$$

and since

$$S_{[-l(n),0]}/l(n) \to \mathbf{E}S(0)$$

a.s. on the event $\{l(\infty) = \infty\}$, we have

$$\begin{aligned} 0 &\leq (Z_{[-n,0]} - Z_{[-n,-l(n),0]})I(Z_{[-n,-l(n),0]} \geq 0)I(l(n) \leq n) \\ &\leq \frac{(S_{[-l(n),0]}/l(n))}{g(n)/l(n))} I(l(n) \leq n) \to 0 \end{aligned}$$

a.s. as n tends to ∞. ∎

Assume now that

$$\mathbf{E}\varphi^{k,i}(0) > 0 \tag{76}$$

for all $1 \leq k, i \leq K$. This property should be understood as some strong connectedness property of the routing mechanism. For $m \geq 0$, let

$$\beta^{i,k}(m) = \min\{j > m : \varphi^{i,k}(-j) > 0\} \tag{77}$$

and

$$\beta^{k}(m) = \max_{i} \beta^{i,k}(m). \tag{78}$$

Corollary 14 (Solidarity property) *Under the condition (76), for each $1 \leq k \leq K$, $n \geq m \geq 0$*

$$(Z_{[-n,0]} - Z^{k}_{-m,[-n,0]})I(\beta^{k}(m) \leq n)I(Z_{-\beta^{k}(m),[-n,0]} \geq 0) \leq S_{[-\beta^{k}(m),0]} \tag{79}$$

a.s. In particular, if $\mathbf{P}(A) = 1$ then

$$\limsup(Z_{[-n,0]} - Z^{k}_{-m,[-n,0]}) \leq S_{[-\beta^{k}(m),0]} \tag{80}$$

a.s. as $n \to \infty$ for all $m \geq 0$ and $k = 1, \ldots, K$ and

$$(Z_{[-n,0]} - Z^{k}_{-m,[-n,0]})/g(n) \to 0 \tag{81}$$

for all $g(n)$ such that $g(n) \to \infty$. In particular,

$$\lim_{n} Z^{k}_{-m,[-n,0]})/n = \gamma \quad a.s. \tag{82}$$

for all k and m, where γ is the constant defined in Theorem 8.

Proof The result follows from the inequality

$$(Z_{-\beta^{k}(m),[-n,0]} - Z^{k}_{-m,[-n,0]})I(\beta^{k}(m) \leq n) \leq 0.$$

■

4.5 Scaling

In what follows, it will be useful to consider various scalings of the arrival processes: for each scaling factor $0 \leq C < \infty$, $-\infty < m \leq n < \infty$, consider the sequences

$$\xi(n, C) = \{C\tau(n), \{\sigma_j^k(n)\}, \{\nu_j^k(n)\}\}, \tag{83}$$

the simple networks

$$\Sigma(n, C) = \{Ct(n), \{\sigma_j^k(n)\}, \{\nu_j^k(n)\}\} \tag{84}$$

and the networks $\Sigma_{[m,n]}(C) = \Sigma(m,C) + \cdots + \Sigma(n,C)$. Let

$$\gamma(C) = \lim_{n\to\infty} Z_{[-n,0]}(C)/n \tag{85}$$

(here $\gamma(1) = \gamma$).

Lemma 5 *$\gamma(C)$ is a continuous and non-increasing function.*

Proof For each $n \geq 0$, $C \geq 0$, $\epsilon \geq 0$

$$Z_{[-n,0]}(C+\epsilon) \leq Z_{[-n,0]}(C) \leq Z_{[-n,0]}(C+\epsilon) + \epsilon(-t(-n)). \quad a.s.$$

So

$$\gamma(C+\epsilon) \leq \gamma(C) \leq \gamma(C+\epsilon) + \epsilon\lambda^{-1}.$$

■

5 Stability Conditions

5.1 Main Ergodic Theorems

For $-\infty < m \leq n < \infty$, let

$$Y_{[m,n]} = Z_{[m,n]}(0) \tag{86}$$

and

$$\rho = \lambda\gamma(0), \tag{87}$$

where the notations are those of the end of § 4.3. We know that

$$\begin{aligned} \gamma(0) &= \lim_{n\to\infty} Y_{[-n,0]}/n = \lim_{n\to\infty} Y_{[-n,-1]}/n \\ &= \lim_{n\to\infty} \mathbf{E}Y_{[-n,0]}/n = \lim_{n\to\infty} \mathbf{E}Y_{[-n,-1]}/n, \quad a.s. \end{aligned}$$

Theorem 10 *If $\rho < 1$, then* $\mathbf{P}(A) = 0$.

Proof For each $l \geq 0$, let N_l be the random variable

$$N_l = \min\{n \geq 0 : Z_{[-n,0]} \geq u(l)\}. \tag{88}$$

If $\mathbf{P}(A) = 1$ then $N_l < \infty$ a.s. for each l. By definition (see (54), for all $n, l \geq 0$, the equalities

$$X_{[-n,l]} = t(l) - t(0) + Z_{[-n,l]} = u(l) + Z_{[0,n+l]} \circ \theta^{-n} \tag{89}$$

hold. Consider the network $\hat{\Sigma}_{[-n,l]}$ obtained by composing the simple networks $\hat{\Sigma}(-n),\ldots,\hat{\Sigma}(l)$, where $\hat{\Sigma}(j)$ has the same service times and switching decisions as $\Sigma(j)$ but an arrival epoch $\hat{t}(j)$ defined as follows:

$$\hat{t}(j) = \begin{cases} t(j) & \text{for } -n \leq j \leq 0, \\ Z_{[-n,0]} & \text{for } 1 \leq j \leq l, \end{cases} \tag{90}$$

For $n \geq N_l$, $t(j) \leq \hat{t}(j)$ for all $-n \leq j \leq l$, so that

$$u(l) + Z_{[-n,l]} \leq Z_{[-n,0]} + Y_{[1,l]},$$

as a direct consequence of the monotonicity property of Lemma 2. Therefore, if $n \geq N_l$,

$$u(l) + Z_{[-n-l,0]} \circ \theta^l \leq Z_{[-n,0]} + Y_{[-l+1,0]} \circ \theta^l, \tag{91}$$

Consider now the network $\tilde{\Sigma}_{[-n,l]}$ defined as above with

$$\tilde{t}(j) = \begin{cases} t(j) & \text{for } -n \leq j \leq 0, \\ t(j) + Z_{[-n,0]} + S_{[1,j-1]} & \text{for } 1 \leq j \leq l, \end{cases} \tag{92}$$

(where $\sum_{i=1}^{0} = 0$). By definition, in network $\tilde{\Sigma}_{[-n,l]}$, all external arrivals taking place later than (and including at) $\tilde{t}(1)$ find an empty system. So, in particular,

$$\tilde{Z}_{[-n,l]} = S(l). \tag{93}$$

The monotonicity property of Lemma 2 also implies that

$$\begin{aligned} X_{[-n,l]} &= u(l) + Z_{[-n-l,0]} \circ \theta^l \\ &\leq \tilde{X}_{[-n,l]} = \tilde{u}(l) + \tilde{Z}_{[-n,l]} \\ &= Z_{[-n,0]} + u(l) + S_{[1,l]}. \end{aligned}$$

So

$$\theta^l Z_{[-n-l,0]} - Z_{[-n,0]} \leq S_{[1,l]} \tag{94}$$

for all $n, l \geq 1$. Combining (94) and (91), we get the inequalities

$$\theta^l Z_{[-n-l,0]} - Z_{[-n,0]} \leq (\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l) + S_{[1,l]} I(n < N_l), \quad a.s. \tag{95}$$

All of random variables in the last inequality are integrable. So

$$\mathbf{E}(\theta^l Z_{[-n-l,0]} - Z_{[-n,0]}) \leq \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l)\} + \mathbf{E}\{S_{[1,l]} I(n < N_l)\}.$$

But

$$\begin{aligned}\mathbf{E}(\theta^l Z_{[-n-l,0]} - Z_{[-n,0]}) &= \mathbf{E}(\theta^l Z_{[-n-l,0]}) - \mathbf{E}(Z_{[-n,0]}) \\ &= \mathbf{E}(Z_{[-n-l,0]}) - \mathbf{E}(Z_{[-n,0]}) = \mathbf{E}(Z_{[-n-l,0]} - Z_{[-n,0]}) \geq 0,\end{aligned}$$

because $Z_{[-n,0]}$ is a non-decreasing sequence. We have

$$\begin{aligned}\mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l))I(n \geq N_l) &= \mathbf{E}(\theta^l Y_{[-l+1,0]} - u(l)) \\ &\quad -\mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l))I(n < N_l)\}.\end{aligned}$$

But

$$\mathbf{E}(\theta^l Y_{[-l+1,0]} - u(l)) = \mathbf{E}(Y_{[-l+1,0]}) - l\lambda^{-1}$$

and since $Z_{[-n,0]} \to \infty$ a.s.,

$$\lim_{n\to\infty} \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l))I(n < N_l)\} = 0$$

and

$$\lim_{n\to\infty} \mathbf{E}\{S_{[1,l]} I(n < N_l)\} = 0.$$

Since

$$0 \leq \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l))I(n \geq N_l)\} + \mathbf{E}\{S_{[1,l]} I(n < N_l)\}$$

for all n, l and since the right-hand side converges to $\mathbf{E}Y_{[-l+1,0]} - l\lambda^{-1}$, we get

$$\mathbf{E}Y_{[-l+1,0]} - l\lambda^{-1} \geq 0 \tag{96}$$

Therefore $\lambda^{-1} \leq \mathbf{E}Y_{[-l+1,0]}/l$ for all $l \geq 1$ and $\lambda^{-1} \leq \lim_{l\to\infty} \mathbf{E}Y_{[-l+1,0]}/l \equiv \gamma(0)$. ∎

Theorem 11 *If $\rho > 1$, then $\gamma(C) > 0$ for all $1 \geq C > 0$. In particular, if $\rho > 1$ then $Z_{[-n,0]} \equiv Z_{[-n,0]}(1) \to \infty$ a.s. as n tends to ∞ (in other words $\mathbf{P}(A) = 1$).*

Proof The monotonicity properties imply

$$Z_{[-n,0]}(C) \geq Z_{[-n,0]} \geq Y_{[-n,0]} + t(-n)$$

a.s. for all $n \geq 0$ and $C \in [0,1]$. So

$$\lim_{n\to\infty}(Z_{[-n,0]}(C))/n \geq \gamma(0) - \lambda^{-1} > 0, \quad a.s.$$

∎

For $C \geq 0$, let

$$\delta(C) = \lim_{n\to\infty}(X_n(C))/n. \tag{97}$$

¿From the definition,

$$\delta(C) = \gamma(C) + C\lambda^{-1}. \tag{98}$$

The monotonicity properties and Lemma 5 imply that $\delta(C)$ is a continuous and non-decreasing function of C.

Remark 14 *The results of Theorems 10-11 and of Theorems 12-14 below are still true for the networks with multi-server stations (see Remark 12).*

5.2 Computation of $\gamma(0)$

Let

$$b = \max_{1\leq k\leq K} b^k. \tag{99}$$

Lemma 6 *In the case $K = 1$, for all $C \geq 0$*

$$\delta(C) = \max(b, Ca) \equiv \max(b^1, Ca). \tag{100}$$

Proof For $K = 1$, the network boils down to a single-server queue with feedback. Since the service times and the switching decisions are associated with stations, the workload in this model is equivalent to that of a single-server queue without feedback and with service times $S^1(n), n \geq 1$. ■

Lemma 7 *For all $C \geq 0$, for each fixed $K \geq 1$*

$$\delta(C) \geq \max(b, Ca). \tag{101}$$

Proof Let k be such that $b^k = b$. Consider an auxiliary sequence of networks $\tilde{\Sigma}_{[m,n]}$ with the same interarrival times and the same switching decisions, and with service times

$$\tilde{\sigma}_j^k = \sigma_j^k \tag{102}$$

for $-\infty < j < \infty$ and

$$\tilde{\sigma}_j^i = 0 \tag{103}$$

for $i = 1, \ldots, K$, $i \neq k$ and $-\infty < j < \infty$.

This boils down to a sequence of single-server queues, and the equality

$$\tilde{\delta}(C) = \max(b, Ca) \tag{104}$$

follows from Lemma 6. The monotonicity properties imply

$$\delta(C) \geq \tilde{\delta}(C) = \max(b, Ca). \tag{105}$$

■

Corollary 15 $\gamma(0) \equiv \delta(0) \geq b$.

Theorem 12 *For all K*

$$\gamma(0) = \delta(0) = b. \tag{106}$$

Proof The proof is given in § 5.3. As it happens, it is simpler to prove a more general result, and we shall in fact prove Theorem 12 for the networks with *bulk arrivals*. ■

Corollary 16 *If $\rho \equiv b/a < 1$ then for each $k \in \{1, \ldots, K\}; l \in \{1, \ldots, K, K+1\}$, the process $\{\Gamma^{k,l}_{[-n,0]}(t), t \geq 0\}$ converges (monotonically increasing) to some limit $\{\Gamma^{k,l}(t)t \geq 0\}$ a.s.*

5.3 Networks with Bulk Arrivals

For each $-\infty < n < \infty$ let

$$\Sigma(n) \equiv \{N(n), T(n), \sigma(n), \nu(n)\} \tag{107}$$

be the composition of $N(n)$ simple networks, where $N(n)$ is a random variable and where $T(n) = (t(n), \ldots, t(n))$, so that all $N(n)$ customers arrive at the same time $t(n)$. For obvious reasons, such a network will be called a *bulk arrival* simple network. For all m, n, the network $\Sigma_{(m,n)}$ is defined as the composition $\Sigma_{(m)} + \ldots \Sigma_{(n)}$ in a way which is totally similar to that of the construction of $\Sigma_{[m,n]}$ (see § 2.1). Our notations will also be similar to those of $\Sigma_{[m,n]}$ (e.g. $Z_{(m,n)}$ for the time to empty the system etc.). In fact, the network $\Sigma_{[m,n]}$ is a particular case of $\Sigma_{(m,n)}$ when $N(i) = 1$ a.s. for $i = m, \ldots, n$.

The stochastic assumptions on bulk arrival networks are slightly different from the preceding case: we assume the sequence

$$\Xi(n) \equiv \{N(n), \tau(n), \sigma(n), \nu(n)\}, \tag{108}$$

(where $\tau(n) = t(n+1) - t(n)$), to be stationary and ergodic, and the random variables

$$\Phi(1), \quad S^k(1) \equiv \sum_{j=1}^{\Phi^k(1)} \sigma_j^k(1) \tag{109}$$

to have the finite first moments. As before, we define $b^k = \mathbf{E}S^k(1)$ and $b = \max_k b^k$. The monotonicity and ergodicity properties of the previous sections are easily extended to this type of networks, and we get in the same way that there exist constants γ and $\gamma(C)$ such that

$$\gamma = \lim \frac{Z_{(-n,0)}}{n}, \quad \gamma(C) = \lim \frac{Z_{(-n,0)}(C)}{n}, \quad \gamma(0) = \lim \frac{Y_{(-n,0)}}{n}, \quad a.s.$$

Theorem 13 *The statement of Theorem 12 holds true for networks with bulk arrivals.*

For the proof, we will assume that $\mathbf{E}\Phi^{0,K} > 0$ (if this is not true the stations should be renumbered) and that $b^k > 0$ for all $k = 1, \ldots, K$ (otherwise we have a model which is equivalent to a network with less than K stations). Our reference network is $\Sigma_{(1,n)}(0)$ everywhere.

For an arbitrary fixed $d > 0$, introduce new sequences of service times:

$$\hat{\sigma}_j^K(l) = \frac{(b+d)\sigma_j^K(l)}{b^K}, \quad \hat{\sigma}_j^k(l) = \frac{b\sigma_j^k(l)}{b^k}, \quad k = 1, \ldots, K-1, \ l = 1, 2, \ldots, j = 1, 2, \ldots$$

Let $\hat{\Sigma}_{(1,n)}(0)$ be the same network as $\Sigma_{(1,n)}$ but with service times $\{\hat{\sigma}_j^k(i)\}$ rather than $\{\sigma_j^k(i)\}$. The monotonicity property implies that $\gamma(0) \leq \hat{\gamma}(0)$. So if we prove the inequality $\hat{\gamma}(0) \leq b + d$, this will complete the proof of the statement of Theorem 13, since $d > 0$ is arbitrary. Till the end of the present subsection, we will work on the networks $\hat{\Sigma}_{(1,n)}(0)$ rather than $\Sigma_{(1,n)}(0)$. For sake of notational simplicity, we will drop the "$\hat{}$" in what follows, which is tantamount to saying that our reference networks $\Sigma(n)$ are such that $b^K = b + d$ and $b^k = b$ for $k = 1, \ldots, K-1$.

The proof is based on the construction of three sequences of auxiliary networks associated with $\{\Sigma_{(n)}\}$.

First auxiliary sequence

For any vector $x = (x_1, \ldots x_l)$, let $R(x)$ be a permutation of the coordinates of x in non-decreasing order, and $\sharp(x) \equiv l$ be the dimension of x. For two vectors $x = (x_1, \ldots, x_l)$ and $y = (y_1, \ldots, y_m)$, (x, y) will denote the vector $(x_1, \ldots, x_l, y_1, \ldots, y_m)$.

Fix n and consider the network $\Sigma_{(1,n)}(0) \equiv (N, T, \sigma, \nu)$. We shall use the notations

$$\Phi^{k,l} \equiv \Phi^{k,l}(1) + \cdots + \Phi^{k,l}(n) \quad \text{and} \quad \Psi_i^{k,l}, 1 \leq i \leq \Phi^{k,l},$$

for the characteristics of $\Sigma_{(1,n)}(0)$. Thus

- $N \equiv N_n = N(1) + \cdots + N(n)$;

- $T \equiv T_n = (T(1), \ldots, T(N))$;
- $\sigma^k \equiv \{\sigma_j^k(1), j = 1, \ldots, \Phi^k(1), \ldots, \sigma_j^k(n), j = 1, \ldots, \Phi^k(n)\}$ is the sequence of service times on station k;
- similar definition for the switching decisions $\{\nu^k\}$.

Let B (resp. D) be the ordered vector whose coordinates are the departure epochs from any station $k = 1, \ldots, K-1$ to station K (resp. from station K to any station $k = 1, \ldots, K$) in $\Sigma_{(1,n)}(0)$. Namely, if

$$\Psi^{k,l} = (\Psi_1^{k,l}, \ldots, \Psi_{\Phi^{k,l}}^{k,l})$$

and

$$\Psi^{\cdot,K} = (\Psi^{1,K}, \ldots, \Psi^{K-1,K}), \quad \Psi^{K,\cdot} = (\Psi^{K,1}, \ldots, \Psi^{K,K}),$$

then

$$\begin{aligned} B &= R(\Psi^{\cdot,K}), \quad \sharp(B) = \sum_{k=1}^{K-1} \Phi^{k,K} \\ D &= R(\Psi^{K,\cdot}), \quad \sharp(D) = \sum_{k=1}^{K} \Phi^{K,k} = \Phi^K - \Phi^{K,K+1}. \end{aligned}$$

Let $\check{\Sigma}_{(1,n)} \equiv (\check{N}, \check{T}, \check{\sigma}, \check{\nu})$ be the following modification of network $\Sigma_{(1,n)}(0)$: at each epoch of $S = R(B, D) \equiv (s_1, s_2, \ldots, s_p)$, such that a customer leaves station k for station l in $\Sigma_{(1,n)}(0)$,

1. the switching is modified in such a way that this customer leaves the network in $\check{\Sigma}_{(1,n)}$ (namely it is routed from k to $K+1$);
2. at the same instant, a new customer is added to the flow from 0 to l.

More precisely, let l_j be the station to which the customer leaving a station at time s_j is sent in $\Sigma_{(1,n)}(0)$. Then

- $\check{N} \equiv \check{N}_n = N + \sharp(B) + \sharp(D)$;
- $\check{T} \equiv \check{T}_n = R(O_N, B, D)$, where O_l denotes the l-dimensional vector with zero-valued coordinates;
- $\check{\sigma} = \sigma$;
- for switching

- $\check{\nu}_j^K = K + 1$ for all $1 \le j \le \Phi^K$;
- $\check{\nu}_j^k = \nu_j^k$ if $\nu_j^k \ne K$ and $\check{\nu}_j^k = K + 1$ if $\nu_j^k = K$, for $k = 1, \ldots, K - 1, j = 1, \ldots, \Phi^k$;
- $\check{\nu}_j^0 = \nu_j^0$, for $1 \le j \le N$ and $\check{\nu}_j^0 = l_{j+N}$, for $N + 1 \le j \le \check{N}$.

∎

Remark 15 *$\Sigma_{(1,n)}(0)$ and $\check{\Sigma}_{(1,n)}$ are equivalent, in that they have the same Γ processes and, therefore, the same residual queue-length and residual service-time processes. Observe in addition that $\check{\Phi}^K = \check{\Phi}^{0,K} = \Phi^K$ and $\check{\Phi}^{0,k} = \Phi^{0,k} + \Phi^{K,k}$ for $k = 1, \ldots, K - 1$.*

Second auxiliary sequence

Associated with $\{\Sigma_{(n)}\}$, define $\tilde{r}(n) = S^K(n)$,

$$v(n) = \sum_{i=1}^{n} \tilde{r}(n), \quad v(0) = 0$$

and

$$\tilde{\Phi}^{0,k}(n) \equiv \Phi^{0,k}(n) + \Phi^{K,k}(n), \quad \forall k = 1, \ldots, K - 1, \quad \tilde{\Phi}^{0,K}(n) = \Phi^K(n).$$

Consider an auxiliary bulk arrival network $\tilde{\Sigma}_{(n)} = (\tilde{N}(n), \tilde{T}(n), \tilde{\sigma}(n), \tilde{\nu}(n))$ with the following characteristics:

- $\tilde{N}(n) = \sum_{k=1}^{K} \tilde{\Phi}^{0,k}(n)$;
- $\tilde{T}(n) = (v(n-1), \ldots, v(n-1), v(n), \ldots, v(n))$, with $v(n-1)$ occurring $\tilde{\Phi}^K(n)$ times and with $v(n)$ occurring $(\tilde{N}(n) - \tilde{\Phi}^K(n))$ times;
- $\tilde{\sigma}(n) = \sigma(n)$;
- for switching:
  - $\tilde{\nu}_j^K(n) = K + 1$, for all $j = 1, \ldots, \Phi^K(n)$;
  - $\tilde{\nu}_j^k(n) = \nu_j^k(n)$ if $\nu_j^k(n) \ne K$ and $\tilde{\nu}_j^k(n) = K + 1$ if $\nu_j^k(n) = K$, for $k = 1, \ldots, K - 1$, $j = 1, \ldots, \Phi^k(n)$.
  - $\tilde{\nu}_j^0(n) = K$ for $1 \le j \le \tilde{\Phi}^{0,K}(n)$ and for $1 \le k \le K - 1$, $\tilde{\nu}_j^0(n) = k$, for all $\tilde{\Phi}^{0,k+1}(n) + \cdots + \tilde{\Phi}^{0,K}(n) + 1 \le j \le \tilde{\Phi}^{0,k}(n) + \cdots + \tilde{\Phi}^{0,K}(n)$.

Thus a bulk of $\tilde{\Phi}^{0,k}(n) \equiv \Phi^{0,k}(n) + \Phi^{K,k}(n)$ customers arrives on station k at time $v(n)$, for $k = 1, \ldots, K-1$, and a bulk of $\tilde{\Phi}^{0,K}(n) \equiv \Phi^{K}(n)$ customers arrives on station K at time $v(n-1)$.

Finally, let $\tilde{\Sigma}_{(1,n)}$ be the composition of the networks $\tilde{\Sigma}_{(i)}, 1 \leq i \leq n$. ∎

Remark 16 *The networks $\tilde{\Sigma}_{(1,n)}$ and $\tilde{\Sigma}_{(1,n)}$ differ in the arrival epochs of external customers only. It follows from the monotonicity property and from Remark 5 that they have the same number of the services on each station and the same total number of services.*

Remark 17 *In $\tilde{\Sigma}_{(1,n)}$, the routes originating from station K are completely disjoint from those originating from $k \in \{1, \ldots, K-1\}$. So the network $\tilde{\Sigma}_{(1,n)}$ can be seen as the union of two disconnected bulk arrival networks, each with jointly stationary driving sequences:*

- *network $\tilde{\Sigma}'_{(1,n)}$, with $K-1$ stations, with bulk interarrival times $\tau'(i) = S^K(i)$ and with service parameters*

$$b^{k'} = E(\sum_{j=1}^{\tilde{\Phi}^k(1)} \sigma_j^k) = E(\sum_{j=1}^{\Phi^k(1)} \sigma_j^k) = b^k, \quad k = 1, \ldots, K-1,$$

  where we used the above remark to get that $b^{k'} = b^k$.

- *network $\tilde{\Sigma}''_{(1,n)}$ with one station K (i.e. a single-server queue with bulk arrivals).*

An important property of the networks $\{\Sigma'_{(1,n)}\}$ and $\{\Sigma''_{(1,n)}\}$ is that each of them satisfies the stationarity and ergodicity assumptions of (108)-(109). In addition, their driving sequences are jointly stationary.

Third auxiliary sequence

Let

$$a_n = \sum_{i=1}^{n} \Phi^{0,K}(i) \equiv \sum_{i=1}^{n} \sum_{j=1}^{N_i} I(\nu_j^0(i) = K) \tag{110}$$

and

$$m_n = \max\{m \geq 1 : F^K(m) \leq a_n/2\}, \tag{111}$$

where $F^K(m) = \Phi^K(1) + \cdots + \Phi^K(m)$ and $m_n = 0$ if $\Phi^K(1) > a_n/2$. Note that that during the time interval $(0, v(m_n))$, station K is never empty in the network $\Sigma_{(1,n)}(0)$.

Let

$$U_n = v(m_n) + \check{Z}_{(1,m_n)} \tag{112}$$

and let α_j^k be the sequence defined as follows:

- for $K = k$:
  - $\alpha_i^K = -\infty$, for $1 \le i \le F^K(m_n)$
  - $\alpha_i^K = U_n$, for $i > F^K(m_n)$;
- for $1 \le k \le K-1$:
  - $\alpha_i^k = S^K(1)$, for $1 \le i \le \Phi^{0,k}(1) + \Phi^{K,k}(1)$;
  - for $j = 1, \ldots, m_n$, $\alpha_i^k = v(j)$, for $\Phi^{0,k}(1) + \Phi^{K,k}(1) + \cdots + \Phi^{0,k}(j-1) + \Phi^{K,k}(j-1) + 1 \le \Phi^{0,k}(1) + \Phi^{K,k}(1) + \cdots + \Phi^{0,k}(j) + \Phi^{K,k}(j)$;
  - $\alpha_i^k = U_n$, for $i > \Phi^{0,k}(1) + \Phi^{K,k}(1) + \cdots + \Phi^{0,k}(m_n) + \Phi^{K,k}(m_n)$.

If we apply this sequence of delays (see Remark 7 and 9) to $\tilde{\Sigma}_{(1,n)}$, we get a network $\check{\Sigma}_{(1,n)}$, which is a majorant of $\tilde{\Sigma}_{(1,n)}$ in that it has the same number of services on each station and the j-th event in station k takes place later in the first than in the second network.

But the Γ-processes of $\check{\Sigma}_{(1,n)}$ and of $\bar{\Sigma}_{(1,n)}$ defined below coincide:

- arrival process
  - customers arrive in bulks of various sizes at the epochs belonging to the set

$$\{0, v(1), \ldots, v(m_n), U_n\},$$

    with a total number of arrivals equal to $\check{N}_n$;
  - $\Phi^K(1)$ customers arrive on station K at time 0, and $\Phi^{0,k}(1) + \Phi^{K,k}(1)$ customers arrive on station k at time $v(1)$, for each $k = 1, \ldots, K-1$;
  - for each $j = 1, \ldots, m_n$ $\Phi^K(j)$ customers arrive on station K at time $v(j-1)$, and $\Phi^{0,k}(j) + \Phi^{K,k}(j)$ customers arrive on station k at time $v(j)$, for each $k = 1, \ldots, K-1$;
  - all other external arrivals take place at time U_n;
- $\bar{\sigma} = \check{\sigma}$;
- $\bar{\nu} = \check{\nu}$;
- on station K, services with indices larger than $F^k(m_n)$ are delayed till U_n;

■

Key relationship between the three networks

Basic features of the auxiliary networks are shown in Fig. 1.

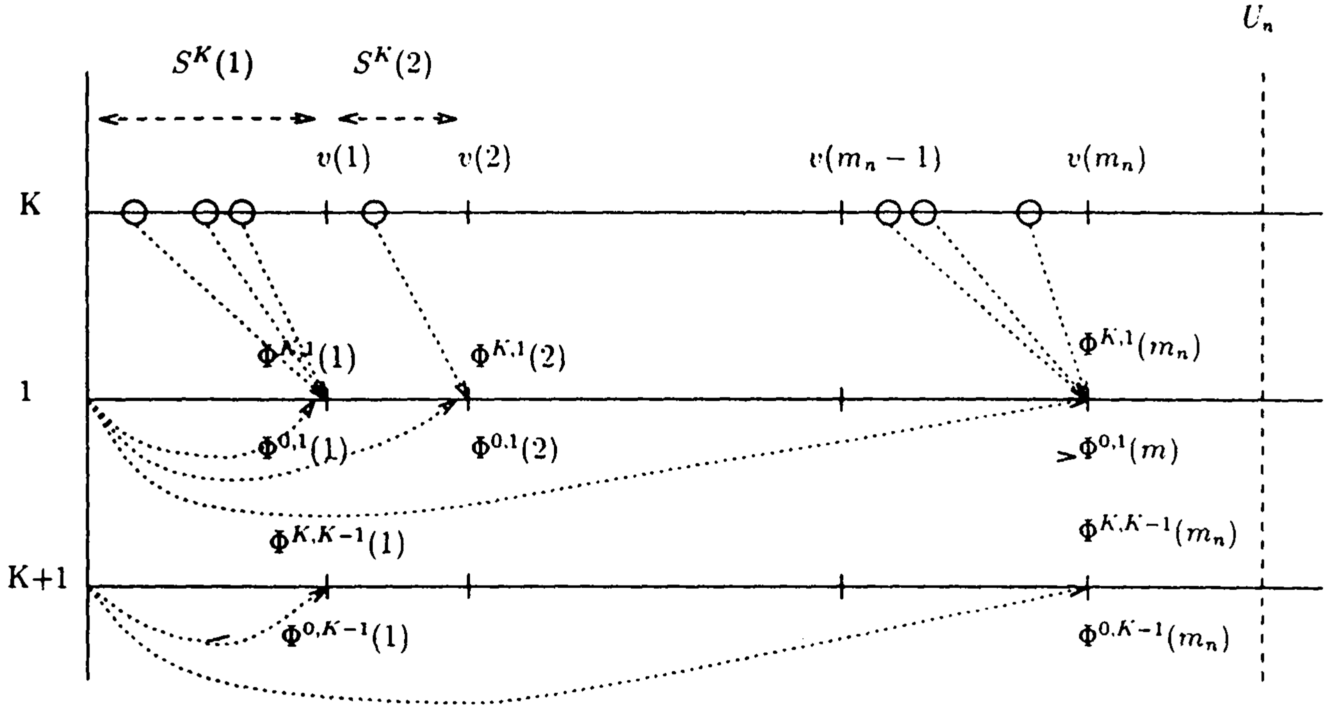

FIGURE 1

Since $\check{\Sigma}_{(1,n)}$ is a majorant of $\bar{\Sigma}_{(1,n)}$ and since

- $\Sigma_{(1,n)}(0)$ and $\bar{\Sigma}_{(1,n)}$ have the same Γ processes.
- $\check{\Sigma}_{(1,n)}$ and $\bar{\Sigma}_{(1,n)}$ have the same Γ processes.

then

$$Y_{(1,n)} \equiv \dot{X}_{(1,n)} \leq \check{X}_{(1,n)} = \dot{X}_{(1,n)}. \tag{113}$$

The key observation is that $\tilde{\Sigma}_{(1,n)}$ is the composition of two networks:

$$\tilde{\Sigma}_{(1,n)} = \tilde{\Sigma}_{(1,m_n)} + \tilde{\Sigma}'_{(m_n+1,n)}, \tag{114}$$

where $\tilde{\Sigma}'_{(m_n+1,n)}$ has the same number of customers, the same service and switching mechanism as $\tilde{\Sigma}_{(m_n+1,n)}$, but all its arrival epochs equal to U_n. By construction, the networks $\tilde{\Sigma}_{(1,m_n)}$ and $\tilde{\Sigma}^1_{(m_n+1,n)}$ are separated (see Remark 10). In addition, $\Sigma_{(m_n+1,n)}(0)$ and $\tilde{\Sigma}'_{(m_n+1,n)}$ have the same Γ-processes. Therefore, we get from (114) that

$$\check{X}_{(1,n)} = \check{X}_{(1,m_n)} + Y_{(m_n+1,n)} = v(m_n+1) + \tilde{Z}_{(1,m_n)} + Y_{(m_n+1,n)} \quad a.s.$$

This together with (113) in turn imply

$$Y_{(1,n)} \leq v(m_n+1) + \tilde{Z}_{(1,m_n)} + Y_{(m_n+1,n)} \quad a.s. \tag{115}$$

Proof of Theorem 13

The proof is by induction on K. For $K = 1$, the result is that of Lemma 6. Assume the theorem holds for networks with $K-1$ stations. Let $\tilde{\lambda}^{-1} = \mathbf{E}\tilde{\tau}_1$. The induction assumption, Remark 17, Corollary 9 and the fact that $\tilde{Z}^K = 0$ allow us to state that if $\tilde{\rho} \equiv \tilde{\lambda} \times \max_{1\leq k\leq K-1} b^k < 1$, then there exists an a.s. finite stationary sequence $\{\hat{Z}(n),\ n \geq 1\}$ such that

$$\tilde{Z}_{(1,n)} \leq \hat{Z}(n), \quad n = 1, 2, \ldots \tag{116}$$

The CLLN implies the relations:

$$\frac{a_n}{n} \to \mathbf{E}\{\Phi^{0,K}(1)\} > 0, \quad m_n \to \infty, \quad a.s.$$

as $n \to \infty$. Similarly,

$$\frac{m}{F^K(m)} \to \frac{1}{\mathbf{E}\Phi^K(1)} < \infty, \quad \frac{m}{F^k(m+1)} \to \frac{1}{\mathbf{E}\Phi^K(1)}, \quad a.s.$$

as $m \to \infty$. Since

$$\frac{m_n}{F^K(m_n)} \geq \frac{m_n}{a_n/2} \geq \frac{m_n}{F^K(m_n+1)},$$

then

$$\frac{m_n}{n} = \frac{m_n}{a_n/2}\frac{a_n/2}{n} \to \frac{\mathbf{E}\Phi^{0,K}(1)}{2\mathbf{E}\Phi^{K}(1)} \equiv c \tag{117}$$

a.s., where $0 < c \leq 1/2$.

We have

$$\frac{v(m_n+1)}{n} \to (b+d)c \quad a.s.$$

In addition, we get from the relation $Y_{(m_n+1,n)}/n = (Y_{(m_n+1,n)}/(n-m_n)) \times ((n-m_n)/n)$ that

$$Y_{(m_n+1,n)}/n \to (1-c)\gamma(0)$$

in probability (see Appendix 8.3). Finally,

$$\frac{\tilde{Z}_{(1,m_n)}}{n} \to 0$$

in probability as $n \to \infty$ (see Appendix 8.3). Therefore (115) implies the inequality:

$$\gamma(0) \equiv \lim_n \frac{Y_{(1,n)}}{n} \leq (b+d)c + (1-c)\gamma(0)$$

for some $0 < c \leq 1/2$. Therefore

$$\gamma(0) \leq b + d. \tag{118}$$

■

6 Stochastic Recursive Sequences

Now we are ready to write down a recursive procedure for constructing the state of the network $\Sigma_{[-n,m+1]}$ from that of $\Sigma_{[-n,m]}$ for each fixed n. More concretely, we want to get a representation of the form

$$W_{[-n,m+1]} = f(W_{[-n,m]}, \eta(m+1)), \tag{119}$$

where the function f is fixed (i.e. non-random and independent of n and m), $\{\eta(m)\}$ is some stationary ergodic sequence and $W_{[-n,m]}$ is the 'state' of network $\Sigma_{[-n,m]}$. Such a representation is often referred to as a stochastic recursive sequence (SRS, see [12]). There are several such representations and we will focus on one of them only.

Consider the space

$$D_+^0(K) \equiv D_+^0 \times D_+^0 \times \cdots \times D_+^0, \quad (K(K+1) \text{ times}), \tag{120}$$

and let

$$W_{[-n,m]} = \{(\Gamma^{1,1}_{[-n,m]}(t), \dots, \Gamma^{1,K+1}_{[-n,m]}(t), \Gamma^{2,1}_{[-n,m]}(t), \dots, \Gamma^{K,K+1}_{[-n,m]}(t)), t \geq 0\} \in D^0_+(K) \tag{121}$$

and

$$\eta(m) = \xi(m), \tag{122}$$

where $\xi(n)$ is defined in (53). Note that $W_{[-n,m]}$ completely defines the service times and the switching decisions of $\Sigma_{[-n,m]}$ after time $t(m)$. So, out of $W_{[-n,m]}$ and $\xi(m+1)$, we can construct all the events of the composition $\Sigma_{[-n,m+1]} = \Sigma_{[-n,m]} + \Sigma_{m+1}$ after $t(m)$. Thus, $W_{[-n,m+1]}$ is a deterministic function f of $W_{[-n,m]}$ and $\xi(m+1)$. The function f is non-decreasing in its first argument. The monotonicity properties, Theorem 10 and Corollary 10 imply the following result:

Theorem 14 *If $\rho < 1$, then for each integer $-\infty < m < \infty$, the sequence $W_{[-n,m]}$ converges monotonically (as $n \to \infty$) a.s. to a finite random variable $W(m) \in D^+_0(K)$, and $\{W(m)\}$ is a stationary ergodic sequence such that*

$$W(m) = W(0) \circ \theta \tag{123}$$

and

$$W(m+1) = f(W(m), \xi(m+1)), \tag{124}$$

for all m. Moreover $\{W(m)\}$ is the minimal stationary solution of the equation (124): if $\tilde{W}(m+1) = f(\tilde{W}(m), \xi(m+1))$ is another stationary solution, then $\tilde{W}(m) \geq W(m)$ $\forall m$ a.s.

Proof The main thing to prove is (124), as the last assertion follows by monotonicity. Observe first that

$$W_{[-n,m+1]} \leq f(\lim_l W_{[-l,m]}, \xi(m+1)) \tag{125}$$

for all n. Therefore

$$W(m+1) \equiv \lim_n W_{[-n,m+1]} \leq f(W(m), \xi(m+1)) \tag{126}$$

In addition, there exists an a.s. finite random number L such that

$$\Gamma^{k,l}_{[-n,m]} = \Gamma^{k,l}_{[-n-1,m]} \tag{127}$$

for all $n > L$ and for each k, l. Therefore the inequality

$$W(m+1) \geq f(W(m), \xi(m+1)) \tag{128}$$

follows immediately from the continuity property (see Corollary 4). ■

Let $\Gamma^{k,l}(m,t)$, $t \geq 0$ be the coordinates of the random variable $W(m)$ and, for each m, k, let $\{Q^k(m,t), t \geq 0\}$ be the associated residual queueing process, $Q(m,t)$ be the process $(Q^1(m,t), \ldots, Q^K(m,t))$ and $\{\chi^k(m,t),\ t \geq 0,\ k = 1, \ldots, K\}$ be the residual service-time process.

Corollary 17 *If $\rho < 1$, then for each m, k, the processes $\{Q^k_{[-n,m]}(t), t \geq 0\}$ converge a.s. (w.r. to the metric d on the space D^0_+) to the process $\{Q^k(m,t), t \geq 0\}$ as $n \to \infty$.*

Corollary 18 *If $\rho < 1$ then for each $-\infty < m < \infty$, $t \geq 0$, the vectors*

$$(Q^1_{[-n,m]}(t), \ldots Q^K_{[-n,m]}(t),\ \chi^1_{[-n,m]}(t), \ldots, \chi^K_{[-n,m]}(t))$$

converge weakly to the vector

$$(Q^1(m,t), \ldots, Q^K(m,t),\ \chi^1(m,t), \ldots, \chi^K(m,t))$$

as $n \to \infty$.

Consider the network $\Sigma = \Sigma(1) + \Sigma(2) + \ldots$, with infinite input sequence $t(1), t(2), \ldots$ Let

$$\tilde{Q}(t) \equiv \{\tilde{Q}^1(t), \ldots, \tilde{Q}^K(t),\ t \geq 0\}$$

be the queue-length process for this network, and for each $n \geq 1$, let

$$\{Q_n(t) \equiv Q(t + t(n)),\ t \geq 0\}$$

be the residual queue-length process. Define now the process

$$Q^{(0)}(t) = \begin{cases} Q(0,t) & \text{for } 0 \leq t < t(1), \\ Q(l, t - t(l)) & \text{for } t(l) \leq t < t(l+1),\ l = 1, 2, \ldots \end{cases} \tag{129}$$

We also define the processes

$$\{Q^{(n)}(t) = Q^{(0)}(t + t(n)), t \geq 0\}, \quad n = 1, 2, \ldots. \tag{130}$$

It follows from Corollary 22 that the sequence $\{Q^{(n)}(t), t \geq 0\}_{n=0}^{\infty}$ is stationary and ergodic (in n).

Corollary 19 *If $\rho < 1$, then*

$$0 \leq Q_n(t) \leq Q^{(n)}(t) \tag{131}$$

a.s. for all $n \geq 0$, $t \geq 0$ and the processes $\{Q_n(t) \circ \theta^{-n},\ t \geq 0\}$ converge monotonically a.s. to the process $Q^{(0)}(t)$.

Proof The proof is similar to that of Corollary 9. ∎

Note that the ergodicity stationarity properties imply the existence of the following limits:

$$\begin{aligned}\lim_t \frac{1}{t}\int_{x=0}^{t} I\{Q^{(0)}(x)\in .\}dx &= \lim_t \frac{1}{t}\int_{x=0}^{t} \mathbf{P}\{Q^{(0)}(x)\in .\}dx \\ &= \frac{\mathbf{E}\{\int_{x=0}^{u(1)} I\{Q^{(0)}(x)\in .\}dx\}}{\mathbf{E}u(1)} \quad a.s. \end{aligned} \tag{132}$$

(where 0/0 means 0, by convention).

Corollary 20 *If* $\rho < 1$, *then*

$$\lim \frac{1}{t}\int_{x=0}^{t} I\{\theta^{-n}\circ Q_n(x)\in .\}dx = \frac{\mathbf{E}\{\int_{x=0}^{u(1)} I\{Q^{(0)}(x)\in .\}\}}{\mathbf{E}u(1)} \quad a.s. \tag{133}$$

as $n \to \infty$.

For each scaling factor $C > 0$, consider the network $\Sigma_{[-n,m]}(C)$. It follows from Theorem 14 that for each $C > b\lambda$, there exists a stationary ergodic sequence $W(m, C)$ such that

$$W(m+1, C) = f(W(m, C), \xi(m+1, C)). \tag{134}$$

In addition, $\{W(m, C)\}$ is the minimal solution of (134).

Theorem 15 *If* $\rho < 1$, *then for each* m

$$W(m, C) \nearrow W(m) \equiv W(m, 1) \tag{135}$$

a.s. as $C \searrow 1$.

Proof The proof is similar to that of Theorem 14. ∎

6.1 General Initial Conditions

The notations are those of § 2.1; we will assume that $t(0) = 0$. Consider an arbitrary network V with a.s. finite input sequence and finite number of services on each station, and such that all arrival epochs are non-positive. For any integer $n \geq 1$, and for any positive real number C, let $_V\Sigma_{[1,n]}(C)$ be the network

$$_V\Sigma_{[1,n]}(C) = V + \Sigma_{[1,n]}(C).$$

We shall say that V is an *initial condition* for $\Sigma_{[1,n]}(C)$ and call the customers of network V a *initial* customers. We shall use the following notations:

- ${}_V\Sigma_{[1,n]} \equiv {}_V\Sigma_{[1,n]}(1)$;
- ${}_V\Gamma^{k,l}_{[1,n]}(C)(t)$, ${}_V W_{[1,n]}(C)$ etc. for the characteristics of ${}_V\Sigma_{[1,n]}(C)$;
- ${}_V\Gamma^{k,l}_{[1,n]}(t)$, ${}_V W_{[1,n]}$ etc. for the characteristics of ${}_V\Sigma_{[1,n]}$.

Lemma 8 *If $\rho < 1$ then for all random initial conditions V and for all real number C such that $b\lambda < C < 1$, one can define an a.s. finite random variable $\beta \equiv \lambda(C, V)$, such that, for all $n \geq 1$,*

$$ {}_V W_{[1,n]} \leq W(n, C), \quad a.s. \tag{136} $$

on the event $\{\beta \leq n\}$, where $W(n, C)$ is the r.v. defined in (134).

Proof Let B_V be the first non-negative time when network V is empty, and let

$$ \beta = \min\{n \geq 1 : (1 - C)t_n \geq B_V\} < \infty \quad a.s. \tag{137} $$

For each $n \geq 1$, consider the network

$$ {}_V\tilde{\Sigma}_{[1,n]}(C) = V + \tilde{\Sigma}_1 + \ldots + \tilde{\Sigma}_n, $$

where $\tilde{\Sigma}_n$ is the same as Σ_n but with arrival epoch

$$ \tilde{t}_i = Ct_i + (1 - C)t_n, \quad i = 1, \ldots, n. \tag{138} $$

The monotonicity property implies that ${}_V\tilde{\Sigma}_{[1,n]}(C)$ is a majorant of ${}_V\Sigma_{[1,n]}$. Note that for $n \geq \beta$ V and $\tilde{\Sigma}_{[1,n]}$ are separated in ${}_V\tilde{\Sigma}_{[1,n]}(C)$ and, therefore,

$$ {}_V W_{[1,n]} \leq {}_V\tilde{W}_{[1,n]}(C) = W_{[1,n]}(C) \leq W(n, C) \tag{139} $$

a.s. on the event $\beta \leq n$. ■

Corollary 21 *If $\rho < 1$, then for each initial condition V*

(i) the sequence $\{{}_V Z_{[1,n]}\}$ is bounded in probability;

(ii) for each $\epsilon > 0$, there exists an element $f \equiv f(\epsilon) \in D^0_+$ such that

$$ \mathbf{P}({}_V\Gamma^k_{[1,n]} \leq f) \geq 1 - \epsilon \tag{140} $$

for all $n \geq 1$, $k = 1, \ldots, K$.

Proof It is enough to prove (i) only. Property (i) follows from the inequality:

$$\sup_n \mathbf{P}(\vee Z_{[1,n]} > x) \leq \max_{1 \leq n \leq N} \mathbf{P}(\vee Z_{[1,n]} > x) + \mathbf{P}(\zeta > N) + \mathbf{P}(Z(1, C) > x), \tag{141}$$

for all $x \geq 0$, $N \geq 1$ and $\lambda b < C < 1$. ■

Remark 18 (Maximal solution) *It is not difficult to see that if $\rho < 1$, then the sequence $\{\tilde{W}(m)\}$ defined by*

- *$\tilde{W}(m)$ is right-continuous a.s.;*
- *$d_K(\tilde{W}(m), \lim_{C \nearrow 1} W(m, C)) = 0$*

forms a maximal stationary solution of (124) (here d_K is a metric in the space $D_+^0(K)$). In particular, $\tilde{W}(m) \leq W(m, C)$ a.s. for all C such that $\lambda b < C < 1$.

7 Coupling-Convergence

Without loss of generality, we can assume that for each n and k, the random variables $\nu_j^k(n)$ and $\sigma_j^k(n)$ are defined for all $j = 1, 2, \ldots$. Let

$$\zeta(n) = \{\tau(n),\ \{\sigma_j^k(n)\}_{j=1}^{\infty},\ k = 1, \ldots, K\}. \tag{142}$$

Consider now the following set of assumptions (referred to as (I) in what follows):

1. $\{\zeta(n)\}_{n=-\infty}^{\infty}$ is a stationary and ergodic sequence;
2. the sequences $\{\{\nu_j^k(n)\}_{j \geq 1},\ k = 0, 1, \ldots, K,\ -\infty < n < \infty\}$ are mutually independent and independent of the sequence $\{\zeta(n)\}_{n=-\infty}^{\infty}$;
3. $\{\nu_j^k(n)\}_{j \geq 1}^{\infty}$ is an i.i.d. sequence, for all $k = 0, 1, \ldots, K; -\infty < n < \infty$.

Remark 19 *Note that under condition (I)*

$$b^k = \mathbf{E}\{\sum_{i=1}^{\varphi^k(1)} \sigma_i^k(1)\} = \sum_{i=1}^{\infty} \mathbf{E}(\sigma_i^k)\mathbf{P}(\varphi^k(1) \geq i). \tag{143}$$

Theorem 16 *Assume that $\rho < 1$. Then, under condition (I),*

(i) the sequence $\{W(n)\}$ is the unique solution of (124);

(ii) one can define all the driving sequences on some probability space in such a way that the sequence $\{_V W_{[1,n]}\}$ coupling-converges to the sequence $\{W(n)\}$, for each initial condition V, i.e.

$$\mathbf{P}\{_V W_{[1,l]} = W(l),\ l = n, n+1, \ldots\} \to 1 \tag{144}$$

as $n \to \infty$.

Proof Note that (ii) implies (i). Indeed, if $\tilde{W}(n)$ is another stationary sequence, then we can consider the initial condition $V = \tilde{W}(0)$ (by this concise notation, we mean any network with the same Γ process as the one generating $\tilde{W}(0)$) and apply (ii). Then $\mathbf{P}\{\tilde{W}(l) = W(l),\ l = n, n+1 \ldots\} \to 1$ as $n \to \infty$. But both of the sequences are stationary, so $\tilde{W}(n) = W(n)$ a.s.

So the only property to prove is (ii). Let

$$p^{k,l} = \mathbf{P}(\nu_1^k(1) = l), \quad k, l = 0, \ldots, K, K+1, \tag{145}$$

where $p^{0,0} = 0$, $p^{K+1,K+1} = 1$ and $p^{K+1,k} = 0$, for $k \le K$. Consider a discrete-time Markov chain $\{R(m), m \ge 0\}$, with state space $\{0, 1, \ldots, K, K+1\}$, with initial value $R(0) = 0$ and with transition matrix $\{p^{k,l}\}$. Let

$$\mu^k = \sharp\{n \ge 1 : R(n) = k\} \tag{146}$$

and

$$\pi^k = \mathbf{E}\mu^k. \tag{147}$$

Note that the random variables μ^k and $\varphi^k(1)$ have the same distribution, so $\pi^k < \infty$ for all $k = 1, \ldots, K$.

In [18]-[19], the following result is proved: for a given transition matrix $\{p^{k,l}\}$, one can define a matrix $\{\tilde{p}^{k,l}\}$ and a renumbering of the state space such that

1. $\tilde{p}^{k,l} = 0$, for all $0 \le l \le k \le K+1$;
2. for all k, l if $\tilde{p}^{k,l} > 0$ then $p^{k,l} > 0$;
3. if $\tilde{R}(n)$ is a Markov chain with initial value $\tilde{R}(0) = 0$ and with transition probabilities $\{\tilde{p}^{k,l}\}$, then

$$\mathbf{P}(\tilde{\mu}^k \ge n) \le \mathbf{P}(\mu^k \ge n), \tag{148}$$

for all $k = 1, \ldots, K$ and $n = 1, 2, \ldots$.

For all $k = 0, 1, \ldots, K$ define the constant

$$C^k = \sup\{C \geq 0 : \quad p^{k,l} \geq C\tilde{p}^{k,l} \quad \forall l = 0, \ldots, K\}. \tag{149}$$

The above results imply that C^k is positive for all k. Finally, for each k, let h^k be a positive number such that $h^k < \min\{1, C^k\}$.

We now return to our network. Our aim is to construct the sequence ν on a specific probability space which is based on the above results. We first construct a sequence of mutually independent r.v.'s

$$\{\tilde{\nu}_j^k(n),\ \hat{\nu}_j^k(n),\ \alpha_j^k(n), \quad k = 0, \ldots, K,\ j = 1, 2, \ldots,\ n = 1, 2, \ldots\}$$

with the following law:

$$\begin{aligned}
\mathbf{P}(\tilde{\nu}_j^k(n) = l) &= \tilde{p}^{k,l}; \\
\mathbf{P}(\alpha_j^k(n) = 1) &= 1 - \mathbf{P}(\alpha_j^k(n) = 0) = h^k; \\
\mathbf{P}(\hat{\nu}_j^k(n) = l) &= \frac{p^{k,l} - h^k \tilde{p}^{k,l}}{1 - h^k}.
\end{aligned}$$

We assume this new sequence to be independent of $\{\zeta(n)\}$. We now choose:

$$\nu_j^k(n) = \alpha_j^k(n)\tilde{\nu}_j^k(n) + (1 - \alpha_j^k(n))\hat{\nu}_j^k(n).$$

For $n = 1, 2, \ldots$, consider the network $\tilde{\Sigma}_{[1,n]}$ with driving sequence $\tilde{\xi}(i) \equiv \{\zeta(i),\ \{\tilde{\nu}_j^k(i)\}\}$ and let

$${}_V\tilde{\Sigma}_{[1,n]} = V + \tilde{\Sigma}_{[1,n]},$$

where V is supposed to be such that ${}_V W \leq W(0, C)$ a.s. Since non-initial customers have acyclic routes and since the traffic intensity is less than 1, then (see [18]) there exists an a.s. finite, positive integer-valued r.v. ζ such that, for all n

$$\tilde{W}_{[1,n]} = {}_V\tilde{W}_{[1,n]} = {}_{W(0,C)}\tilde{W}_{[1,n]} \quad a.s. \tag{150}$$

on the event $\{n \geq \zeta\}$. Let

$$L = \min\{n \geq 1 : \quad \mathbf{P}(\zeta = n) > 0\}.$$

Then the event

$$A \equiv \{\zeta = L\} \bigcap \{\alpha_j^k(i) = 1,\ i = 1, \ldots, L,\ k = 0, \ldots, K,\ j = 1, \ldots, \tilde{\varphi}^k(i)\}$$

has a positive probability. Therefore the events $\{\{\zeta \leq n\} \bigcap \{\theta^n A\}, \quad n = 1, 2, \ldots\}$ form a sequence of renovating events for the sequence $\{{}_V W_{[1,n]}\}$, and the statement of the theorem follows from Theorem 3 of [12] or of [18]. ∎

8 Appendix

8.1 Appendix 1: The Geometry of Routes

Case $N = 1$

Lemma 9 *Let $r = (r_1, \ldots, r_\varphi, K+1)$ be a successful route and G be a G.O.D.G. generated by r. Choose $k \in \{1, \ldots, K\}$ such that $\varphi^k > 0$. Consider a path $\tilde{r}$ starting from k, $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_m)$. Then $\tilde{r}$ is an admissible route.*

Proof If $k = r_1$ then $\tilde{r} = r$ and the statement is obvious. Assume that $k \neq r_1$. We use the notations of Procedure 1 and refer to an object associated with $G(t)$ by adding the argument (t). We have $c^l(1) = d^l(1)$ for each node $l = 1, \ldots, K$ different from r_1, while $d^{r_1}(1) = c^{r_1}(1) + 1$. More generally, for all $t \geq 2$, if $n(t) \neq K+1$, then $c^l(t) = d^l(t)$ for each node $l = 1, \ldots, K$ not belonging to $\{r_1, k, n(t)\}$, while $d^{r_1} = c^{r_1} + 1$, $d^k = c^k - 1$ and $d^{n(t)} = c^{n(t)} + 1$. Assume that the path originating from k ends in node $l \neq K+1$. This means that there exists an integer t such that $d^{n(t)} = 0$. In view of the above relations, $l = k$ necessarily.

Lemma 10 *Assume the path $\tilde{r}$ of length m to be a simple circuit (i.e. $\tilde{r}_i \neq \tilde{r}_j$ for all $1 \leq i, j < m$, $i \neq j$ and $\tilde{r}_m = \tilde{r}_1$). Then we can find $l \in \{1, \ldots, K\}$ such that:*

- *$\tilde{r}_i = l$ for some $i \in \{1, \ldots, m-1\}$ (i.e. l belongs to the path $\tilde{r}$);*
- *the route r can be represented under the form:*

$$r = (r_1, \ldots, r_a, \ldots, r_b, \ldots, r_\varphi, K+1)$$

where $r_a = r_b = l$ and

$$(r_a, \ldots, r_b) = (\tilde{r}_i, \tilde{r}_{i+1}, \ldots, \tilde{r}_{m-1}, \tilde{r}_1, \ldots, \tilde{r}_{i-1}, \tilde{r}_i).$$

Proof Let A be the set of nodes in the sequence $\{\tilde{r}_i\}_{i=1}^{m-1}$. Let $a = \min\{n : r_n \in A\}$ and $l = r_a = \tilde{r}_i$. ¿From the definition of a path, $r_{a+j} = \tilde{r}_{i+j}$, for $0 \leq j \leq m-1-i$ and $r_{a+j} = \tilde{r}_{1+i-m}$ for $m-i \leq j \leq m-1$. ■

Corollary 22 *Let $\mathcal{A}$ be the sequence of arcs associated with the path $\tilde{r}$ of Lemma 10. The O.D.G. $\dot{G}$ obtained from G by removing the arcs of $\mathcal{A}$ is a G.O.D.G. generated by the route*

$$\dot{r} = (r_1, \ldots, r_a, r_{b+1}, \ldots, r_\varphi, K+1).$$

Lemma 11 *Let $\tilde{r}$ be a path such that $\tilde{r}_i \neq \tilde{r}_j$ for all $1 \leq i, j < m$ and $\tilde{r}_m = K + 1$. Then we can find an integer a such that*

$$\tilde{r} = (r_a, r_{a+1}, \ldots, r_\varphi, K + 1).$$

In addition, the O.D.G. $\hat{G}$ obtained from G by removing the arcs of $\mathcal{A}$ is generated by the route

$$\hat{r} = (r_1, \ldots, r_a).$$

Proof Define A, a and l as in the proof of Lemma 10. From the definition of the path, $r_{a+j} = \tilde{r}_{i+j}$ for $0 \leq j \leq m - i$. In particular, $r_{a+m-i} = \tilde{r}_m = K + 1$. But $\varphi^k > 0$. So $l = k$ and $\tilde{r}_1 = r_a$. The fact that r is a circuit follows from Lemma 9. ■

Lemma 12 *Let $\tilde{r}$ to be a circuit (i.e. a path such that $\tilde{r}_m = \tilde{r}_1$). Then we can find finite integers l and*

$$a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_l < b_l < \varphi$$

such that $r_{a_i} = r_{b_i}$ and such that the graph $\hat{G}$ obtained from G by removing the arcs of the path $\tilde{r}$ is a G.O.D.G. generated by the route

$$\hat{r} \equiv (r_1, \ldots, r_{a_1}, r_{b_1+1}, \ldots, r_{a_2}, r_{b_2+1}, \ldots, r_{a_l}, r_{b_l+1}, \ldots, r_\varphi, K + 1). \tag{151}$$

See Figure 2 for an example.

Proof The proof is by induction on the length m of the path. For $m = 2$, the statement is clear, because circuits of length 2 are necessarily simple. Assume we proved the property for all m_0 with $1 \leq m_0 < m$. If $\tilde{r}$ is a simple circuit, then the statement is true from Lemma 10 and Corollary 22. Otherwise let α and β be the smallest integers such that $1 \leq \alpha < \beta$ and such that $(\tilde{r}_\alpha, \tilde{r}_{\alpha+1}, \ldots, \tilde{r}_\beta)$ is a simple circuit; the circuit $\tilde{r}' \equiv (\tilde{r}_\alpha, \tilde{r}_{\alpha+1}, \ldots, \tilde{r}_\beta)$ is necessarily a path of G. Let G' be the graph defined as in Corollary 22 but when removing the arcs associated with the path $\tilde{r}'$ of G. From Corollary 22, $\hat{G}'$ is a G.O.D.G. generated by a route of the form

$$\hat{r}' \equiv (r'_1, \ldots, r'_{\varphi'}, K + 1) = (r_1, \ldots, r_a, r_{b+1}, \ldots r_\varphi, K + 1). \tag{152}$$

where $r_a = r_b$. Corollary 22 also implies that $\tilde{r}'' \equiv (\tilde{r}_1, \ldots, \tilde{r}_\alpha, \tilde{r}_{\beta+1}, \ldots, \tilde{r}_m)$ is a path of length $m'' < m$ for the G.O.D.G. $\hat{G}'$. ¿From the induction assumption, when removing the arcs of $\tilde{r}''$ from $\hat{G}'$, we get a G.O.D.G. generated by a route $\hat{r}''$ of the form given in (151), namely.

$$\hat{r}'' \equiv (r'_1, \ldots, r'_{a'_1}, r'_{b'_1+1}, \ldots, r'_{a'_2}, r'_{b'_2+1}, \ldots, r'_{a'_l}, r'_{b'_l+1}, \ldots, r'_{\varphi'}, K + 1), \tag{153}$$

where $r'_{a'_i} = r'_{b'_i}$ for all i. Equations (152)-(153) show that the property holds for all circuits of length m. ■

Lemma 13 *Let $\tilde{r}$ be a path such that $\tilde{r}_m = K + 1$. We can find finite integers l and a_i, b_i, $i = 1, \ldots, l$ such that*

$$r_{a_i} = r_{b_i}; i = 1, \ldots, l - 1.$$

and such that the graph $\hat{G}$ is generated by the route

$$\hat{r} \equiv (r_1, \ldots, r_{a_1}, r_{b_1+1}, \ldots, r_{a_2}, r_{b_2+1}, \ldots, r_{a_l}).$$

In general, this sequence is not an admissible route. See Figure 3 for an example.

Proof The proof follows from Lemmas 11-12 and from induction arguments. ∎

Case $N = 2$

Lemma 14 *Theorem 1 holds true for $N = 2$.*

Proof Let $(G, \{T^k\}_{k=1}^K)$ be a G.O.D.G. generated by the sequence of routes $R = (r(1), r(2))$, where $r_1(1) = l_1; r_1(2) = l_2$. Assume that token 1 is on node l_1 and token 2 on node l_2. Let route $r(1)$ be associated with token 1 and $r(2)$ with token 2. We construct two new routes $\tilde{r}(1)$ and $\tilde{r}(2)$ which generate the same G.O.D.G. and which are associated with tokens 2 and 1, respectively. If $l_1 = l_2$ then we take $\tilde{r}(1) = r(1)$ and $\tilde{r}(2) = r(2)$. Otherwise let $L = (L_1, \ldots, L_m)$ be the path of G originating from node $L_1 = l_2$. Since $d^k = c^k$ for $k \neq l_1, l_2$ and $d^k = c^k + 1$ for $k \in \{l_1, l_2\}$, then $L_m \neq k$ for all $k \leq K$. So $L_m = K + 1$ necessarily. We take $\tilde{r}(1) = L$. In order to define $\tilde{r}(2)$, consider the last arc (L_{m-1}, L_m) of L. There are two possibilities:

- **(a) This arc belongs to the route r(2).**

  If the arc (L_1, L_2) belongs to $r(1)$, Lemma 10 implies that we can find an integer $q \geq 2$ such that $L_q = L_1$ and all the arcs $(L_1, L_2), (L_2, L_3), \ldots, (L_{q-1}, L_q)$ belong to $r(1)$.

  Similarly, if for some $p \geq 2$, (L_{p-1}, L_p) belongs to $r(2)$ and (L_p, L_{p+1}) belongs to $r(1)$, we can then find an integer $q \geq p + 1$ such that the arcs $(L_p, L_{p+1}), (L_{p+1}, L_{p+2}), \ldots, (L_{q-1}, L_q)$ all belong to $r(1)$ and $L_p = L_q$.

  So the sequence of arcs associated with path L is composed of circuits of arcs which all belong to $r(1)$ and of certain sequences of arcs belonging to $r(2)$.

  We now prove that all the arcs belonging to $r(2)$ also belong to L. Consider the first arc of $r(2)$. If it does not belong to (the sequence of arcs associated with) L, what precedes implies that we have to return to L_1 infinitely often; since $\varphi(1)$ is finite, this is not possible. Thus, the first arc of $r(2)$ belongs to L. The same argument is applicable (by induction) to all arcs belonging to $r(2)$.

Consider the O.D.G. $\hat{G}$ on $\{1, \ldots, K+1\}$ defined by the set of arcs of G which do not belong to the path $\check{r}(1)$. Thus the set of arcs of $\hat{G}$ consists of all arcs of route $r(1)$ but for a finite number of circuits. It follows from Lemma 12 and from an immediate induction argument that $\hat{G}$ is a G.O.D.G. generated by some successful route $\tilde{r}(2)$ originating from node l_1.

- **(b) This arc belongs to route $r(1)$.**

  If the path L only consists of arcs which belong to $r(1)$, then Lemma 13 shows that $\check{r}(1)$ is a route originating from node l_1 and ending in node l_2 (since L is the a circuit from l_2 to l_2). Let $\tilde{r}(2)$ be the concatenation of $\check{r}(1)$ and $r(2)$. The sequences $\tilde{r}(1)$ and $\tilde{r}(2)$ are successful route which generate the G.O.D.G. G by construction.

  Assume now that there exists an integer $q \geq 2$ such that (L_{q-1}, L_q) belongs to $r(2)$ and (L_{q+i}, L_{q+i+1}) to $r(1)$, for all $i \geq 0; i < m-q$. Since the arcs (L_{j-1}, L_j) belong to $r(1)$, they form a finite number of circuits, as it was shown in (a); thus the set of the arcs which belong to $r(1)$ and not to L is generated by some route $\check{r}(1) \equiv (\check{r}_1(1), \ldots, \check{r}_n(1))$ the form of which is given by Lemma 13. We have in particular $\check{r}_1(1) = l_1$ and $\check{r}_n(1) = L_q$.

  Concerning the arcs opf L which belong to $r(2)$, as in case (a), simple induction arguments show that we can find a number p such that all the arcs $(r_1(2), r_2(2)), \ldots, (r_{p-1}(2), r_p(2))$ (and only these arcs of $r(2)$) belong to L. Moreover, $r_p(2)$ has to be equal to L_q.

  We take $\tilde{r}(2) = (\check{r}_1(1), \ldots, \check{r}_n(1), r_{p+1}(2), \ldots, r_{\varphi(2)}(2), K+1)$.

∎

Proofs of Theorems

Proof of Theorem 1

Fix some token j. We shall show that if there exists a generator $R = (r(1), \ldots, r(N))$ such that token j is associated with route $r(n)$ (where $1 < n \leq N$) then there exists a generator $\tilde{R} = (\tilde{r}(1), \ldots, \tilde{r}(N))$ such that token j is associated with route $\tilde{r}(n-1)$. The statement of Theorem 1 follows by induction. We define $\tilde{R}$ as follows:

- take $\tilde{r}(j) = r(j)$ for all $j \neq n, j \neq n-1$;
- consider the G.O.D.G. G' generated by $(r(n-1), r(n))$ and use Lemma 14.

■

Proof of Theorem 2

The proof is by induction on φ and N. If $\varphi = N$, then the statement is clear.

Assume that for $\varphi < \varphi_0$, Theorem 2 was proved for each G.O.D.G. with no more than N tokens. We prove it also holds true for a G.O.D.G. with N tokens and with $\varphi = \varphi_0$ arcs.

If $X(1) = j$ for some token j, then consider a generator $R = (r(1), \ldots, r(N))$ such that token j is associated with route $r(1)$ (this is possible from Theorem 1). So $G(2)$ is a G.O.D.G. generated by the sequence of successful routes

$$\tilde{R} = (\tilde{r}(1), r(2), \ldots, r(N))$$

($\tilde{R} = (r(2), \ldots, r(N))$ if $\tilde{r}(1) = \emptyset$) , where

$$\tilde{r}(1) = (r_2(1), \ldots, r_{\varphi(1)}(1), K+1)$$

($\tilde{r}(1) = \emptyset$ if $\varphi(1) = 1$).

Since the number of arcs in $G(1)$ is equal to $\varphi - 1$, the proof follows by induction. ■

Proof of Theorem 3

The proof is by induction. Let $P = \sharp(\mathcal{A})$. If $P = N$, then the statement is clear, whatever the value of N. Assume it holds true for $P < l$ and for all $N \leq N_0$.

If $G(2)$ consists of $G \equiv G(1)$ with the arc from node k_1 to node k_2, (where $k_2 \leq K$) removed, then we can apply the induction step to the O.D.G.$G(2)$ with N tokens. Since $H(2) \equiv H(G(2), X)$ is a G.O.D.G., we can construct (see Theorem 2.1) a generator $R = (r(1), \ldots, r(N))$ for $H(2)$, where route $r(1)$ is associated with the token we moved. Therefore $(H, \{T^k\}_{k=1}^K)$ is a G.O.D.G. with a generator

$$\tilde{R} = (\tilde{r}(1), r(2), \ldots, r(N)),$$

where $\tilde{r}(1) = (k_1, r_1(1), \ldots, r_{\varphi(1)}(1), K+1)$.

If this arc is from node k_1 to node $(K+1)$ then we can apply the induction step to the O.D.G. $G(2)$ with $(N-1)$ tokens and construct a generator $R = (r(2), \ldots, r(N))$ for $H(2)$. So $(H, \{T^k\}_{k=1}^K)$ is a G.O.D.G. with generator

$$\tilde{R} = (r(1), r(2), \ldots, r(N)),$$

where $r(1) = (k_1, K+1)$. Thus, in both cases, the result holds true for $P = l$ and for $N = N_0$. ■

8.2 Appendix 2: Short Proof of the Conservation Rule

Our departure point is the set of recursive equations (15). Let

$$\Phi_N^k = \sup\{j \geq 1 \ : \ \Psi_j^k < \infty\}.$$

The conservation rule states that $\Phi_N = (\Phi_N^1, \ldots, \Phi_N^K)$, which could be a function of

$$\{\{t(j)\}_{j=1}^N,\ \{\sigma_j^k\},\ \{\nu_j^k\},\ j \geq 1,\ k = 1, \ldots, K\}$$

is actually a function of

$$\nu \equiv \{\{\nu_j^k\},\ j \geq 1,\ k = 1, \ldots, K\}$$

only.

In view of the basic monotonicity property, there exists a positive real number x such that the network $\tilde{\Sigma}_N$ with driving sequence

$$\{\tilde{t}(j)\}_{j=1}^N,\ \{\sigma_j^k\},\ \{\nu_j^k\},\ j \geq 1,\ k = 1, \ldots, K\},$$

where $\tilde{t}_j = t_j + jx$, is fully separated (i.e. each external customer finds an empty system upon its arrival). It is clear that $\tilde{\Phi}_N^k$ only depends on ν. Therefore, the conservation rule will be proved if we show that

$$\tilde{\Phi}_N^k = \Phi_N^k, \quad \forall k. \tag{154}$$

We have

$$\tilde{\Phi}_N^k \leq \Phi_N^k, \quad \forall k.$$

Indeed, since $t(j) \leq \tilde{t}(j)$, for all j, it follows from monotonicity that $\tilde{\Psi}_j^k \leq \Psi_j^k$, $\forall j, k$, which concludes the proof.

We now prove that

$$\tilde{\Phi}_N^k \geq \Phi_N^k, \quad \forall k.$$

¿From the very definition of $\tilde{\Phi}$, for all $A > 0$,

$$\tilde{\Psi}^k_{\tilde{\Phi}_N^k+1} > A.$$

Thus, from Corollary 2 (which can be proved directly from (15)), we get that

$$\tilde{\Psi}^k_{\tilde{\Phi}_N^k+1} \leq Nx + \Psi^k_{\tilde{\Phi}_N^k+1}.$$

Therefore, for all $A \geq 0$,

$$\Psi^{k}_{\hat{\Phi}^{k}_{N}+1} + Nx \geq A$$

which implies that $\Psi^{k}_{\hat{\Phi}^{k}_{N}+1} = \infty$. ∎

8.3 Appendix 3

Proof of

$$\frac{Y_{(m_n+1,n)}}{n - m_n} \to \gamma(0) \quad \text{in probability.}$$

For $0 < \delta < c$, let

$$x_n = [(c-\delta)n]; \quad y_n = [(c+\delta)n] + 1,$$

where $[x]$ denotes the integer part of x. Let $H_n = Y_{(m_n+1,n)}/(n - m_n) - \gamma(0)$. For all $\epsilon > 0$

$$\mathbf{P}(|H_n| > \epsilon) = \mathbf{P}(H_n > \epsilon) + \mathbf{P}(H_n < -\epsilon)$$

and for all $0 < \delta < \epsilon(1-c)/(2\gamma(0)+\epsilon)$

$$\mathbf{P}(H_n > \epsilon) \leq \mathbf{P}(|\frac{m_n}{n} - c| > \delta) + \mathbf{P}(\frac{Y_{(x_n,n)}}{n - y_n} > \gamma(0) + \epsilon).$$

The last expression tends to 0 as n goes to ∞, as it can be seen from the following relations:

$$\begin{aligned}
\frac{Y_{(0,n-x_n)}}{n - x_n} &\to \gamma(0); \\
\frac{n - x_n}{n - y_n} &\to \frac{1 - c + \delta}{1 - c - \delta}; \\
\gamma(0)\frac{1 - c + \delta}{1 - c - \delta} &= \gamma(0)\frac{1 + 2\delta}{1 - c - \delta} < \gamma(0)(1 + \frac{\epsilon}{\gamma(0)}) = \gamma(0) + \epsilon.
\end{aligned}$$

$\mathbf{P}(H_n < -\epsilon) \to 0$, by similar arguments. ∎

Proof of

$$\tilde{Z}_{(1,m_n)}/n \to 0 \quad \text{in probability.}$$

Note that $\tilde{Z}_{(1,m)} \leq \tilde{Z}(m)$ a.s. for each m (see (116) and

$$\tilde{Z}_{(1,m+l)} \leq \tilde{Z}_{(1,m)} + \tilde{Z}_{(m+1,m+l)} \leq \tilde{Z}(m) + \sum_{j=m+1}^{m+l} \tilde{Z}_{(j)}.$$

where $\tilde{Z}_{(j)}$ is a stationary and ergodic sequence with finite first moment $\mathbf{E}\tilde{Z}_{(1)} \equiv h$.

Therefore for all $\epsilon > 0$, $0 < \delta < \min(c, \epsilon/4h)$

$$\begin{aligned}
\mathbf{P}(\tilde{Z}_{(1,m_n)}/n > \epsilon) &\leq \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\max_{x_n \leq l \leq y_n} \tilde{Z}_{(1,l)}/n > \epsilon) \\
&\leq \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\{\tilde{Z}(x_n) + \sum_{j=x_n+1}^{y_n} \tilde{Z}_{(j)}\}/n > \epsilon) \\
&\leq \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\tilde{Z}(x_n) > \epsilon/2) + \mathbf{P}(\sum_{j=1}^{y_n - x_n} \tilde{Z}_{(j)} > \epsilon/2).
\end{aligned}$$

The last expression tends to 0 as n goes to ∞ because

$$\begin{aligned}
\mathbf{P}(\tilde{Z}(x_n) > \epsilon/2) = \mathbf{P}(\tilde{Z}(1) > n\epsilon/2) &\rightarrow 0; \\
(\sum_{j=1}^{y_n - x_n} \tilde{Z}_{(j)})/(y_n - x_n) &\rightarrow h; \\
\frac{y_n - x_n)}{n} &\rightarrow 2\delta < \epsilon/2h.
\end{aligned}$$

■

Future Work

The consequences of the construction that is proposed here will be investigated in a companion paper which will primarily focus on the i.i.d case. We would also like to mention that these techniques extend almost directly to the class of Petri nets defined in [5] (see [7]).

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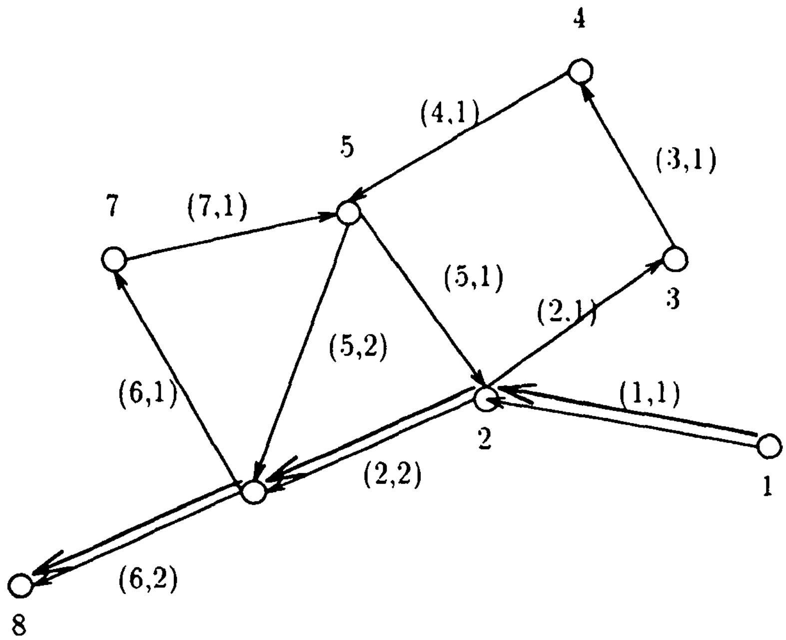
Route generating G: R=(1,2,3,4,5,2,6,7,5,6,8)

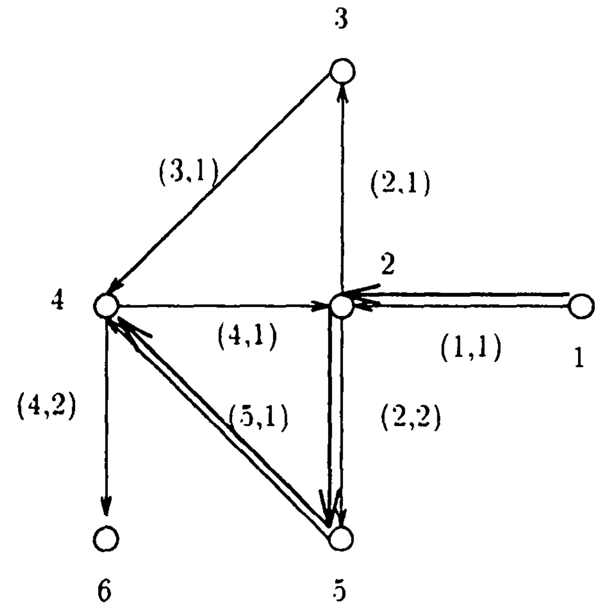


Path originating from node 5 : $\tilde{r} = (5, 2, 3, 4, 5, 6, 7, 5)$, a circuit

$\hat{G}$ **is generated by the route** $\hat{r} = (1, 2, 6, 8)$ (in bold)

FIGURE 2

Route generating the G.O.D.G :r=(1,2,3,4,2,5,4,6)



Path originating from node 4: $\tilde{r} = (4, 2, 3, 4, 6)$

Graph $\hat{G}$ generated by $\hat{r} = (1, 2, 5, 4)$ (bold)

FIGURE 3

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